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Infrared structure of $e^+e^- \to 3$ jets at NNLO

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Abstract: We describe the calculation of the next-to-next-to-leading order (NNLO) QCD corrections to three-jet production and related event shape observables in electron-positron annihilation. Infrared singularities due to double real radiation at tree level and single real radiation at one loop are subtracted from the full QCD matrix elements using antenna functions, which are then integrated analytically and added to the two loop contribution. Using this antenna subtraction method, we obtain numerically finite contributions from five-parton and four-parton processes, and observe an explicit analytic cancellation of infrared poles in the four-parton and three-parton contributions. All contributions are implemented in a flexible parton-level event generator programme, allowing the numerical computation of any infrared-safe observable related to three-jet final states to NNLO accuracy.

Keywords: QCD, Jets, LEP HERA and SLC Physics.

Contents

1. Introduction

Among jet observables, the three-jet production rate in electron-positron annihilation plays a very prominent role. The initial experimental observation of three-jet events at PETRA [1], in agreement with the theoretical prediction [2], provided first evidence for the gluon, and thus strong support for the theory of Quantum Chromodynamics (QCD) [3]. Subsequently the three-jet rate and related event shape observables were used for the precise determination of the QCD coupling constant α_s (see [4] for a review). Especially at LEP, three-jet observables were measured to a very high precision and the error on the extraction of α_s from these data is dominated by the uncertainty inherent in the theoretical description of the jet observables. This description is at present based on a next-to-leading order (NLO) calculation [5 – 10], combined with next-to-leading logarithmic (NLL) resummation [11, 12] and inclusion of power corrections [13]. The calculation of next-to-next-to-leading order (NNLO), i.e. $\mathcal{O}(\alpha_s^3)$, corrections to the three-jet rate in e^+e^- annihilation has therefore been high on the list of priorities for a long time [14].

Besides its phenomenological importance, the three-jet rate has also served as a theoretical testing ground for the development of new techniques for higher order calculations in QCD: both the subtraction [5, 15, 16] and the phase-space slicing [7] methods for the extraction of infrared singularities from NLO real radiation processes were developed in the context of the first three-jet calculations. The systematic formulation of phase-space slicing [9] as well as the dipole subtraction [10] method were also first demonstrated for three-jet observables, before being applied to other processes.

Over the past years, many of the ingredients necessary for NNLO calculations of jet observables have become available: two-loop corrections to all phenomenologically relevant massless $2 \to 2$ [17] and $1 \to 3$ [18, 19] reactions were computed already several years ago, while one-loop $2 \to 3$ [20] and $1 \to 4$ [21] matrix elements are available for even longer. Despite all ingredients being available in principle, until recently it was not possible to perform NNLO calculations of any kind of exclusive observables, since techniques for the extraction of multiple real radiation singularities at NNLO were not sufficiently developed.

Up to now, the only general method for handling this problem was the sector decomposition technique [23, 22] for the treatment of real radiation singularities. Using this, NNLO calculations of exclusive processes were performed for $e^+e^- \to 2j$ [24], Higgs production [25] and vector boson production [26] at hadron colliders, as well as the QED corrections to muon decay [27].

Furthermore, exploiting the specific kinematic features of the observable under consideration, exclusive NNLO results were derived for the forward-backward asymmetry in e^+e^- annihilation [28], for $e^+e^- \to 2j$ [29, 30] and most recently for Higgs production at hadron colliders [31].

In the present paper, we employ a recently developed general technique for the treatment of infrared singularities, antenna subtraction [32], to derive the NNLO corrections to three-jet production in electron-positron annihilation. The first phenomenological applications of our results to the thrust distribution were documented earlier in [33].

The paper is structured as follows: in section 2, we outline the perturbative calculation of jet observables and summarise the antenna subtraction method used here. The implementation of this method requires phase space mappings, which are described in section 3. All relevant tree-level, one-loop and two-loop matrix elements are listed in section 4. Section 5 briefly summarises how the NLO corrections are implemented using antenna subtraction. Sections 6–12 contain the subtraction terms constructed for all colour factors relevant in this calculation. The numerical implementation of all terms into a parton-level event generator is described in section 13. As a first example of the implementation, we discuss the NNLO corrections to the thrust distribution in section 14. A summary and an outlook on applications is given in section 15.

2. Perturbative calculation of jet observables in e^+e^--annihilation

To obtain the perturbative corrections to a jet observable at a given order, all partonic multiplicity channels contributing to that order have to be summed. In general, each partonic channel contains both ultraviolet and infrared (soft and collinear) singularities. The ultraviolet poles are removed by renormalisation, however for suitably inclusive observables, the soft and collinear infrared poles cancel among each other when all partonic channels are summed over [34].

While infrared singularities from purely virtual corrections are obtained immediately after integration over the loop momenta, their extraction is more involved for real emission (or mixed real-virtual) contributions. Here, the infrared singularities only become explicit after integrating the real radiation matrix elements over the phase space appropriate to the jet observable under consideration. In general, this integration involves the (often iterative) definition of the jet observable, such that an analytic integration is not feasible (and also not appropriate). Instead, one would like to have a flexible method that can be easily adapted to different jet observables or jet definitions.

Three types of approaches for this task have been developed so far. Phase-space slicing techniques [7, 9, 35] decompose the final state phase space into resolved regions, which are integrated numerically and unresolved regions, which are integrated analytically. The sector decomposition approach [23, 22] divides the integration region into sectors containing a single type of singularity each. Subsequently, the phase space integration is expanded into distributions. In this approach, the coefficients of all infrared divergent terms, as well as the finite remainder, can be computed numerically. Finally, subtraction methods [5, 15, 16] extract infrared singularities of the real radiation contributions using infrared subtraction terms. These terms are constructed such that they approximate the full real radiation matrix elements in all singular limits while still being simple enough to integrate analytically.

To specify the notation, we define the tree-level n-parton contribution to the m-jet cross section (for tree level cross sections $n = m$; we leave $n \neq m$ for later reference) in d dimensions by,

$$\mathrm{d}\sigma^B = \mathcal{N} \sum_n \mathrm{d}\Phi_n(p_1, \ldots, p_n; q) \frac{1}{S_n} |\mathcal{M}_n(p_1, \ldots, p_n)|^2 \, J_m^{(n)}(p_1, \ldots, p_n). \tag{2.1}$$

the normalisation factor $\mathcal{N}$ includes all QCD-independent factors as well as the dependence on the renormalised QCD coupling constant α_s, $\sum_n$ denotes the sum over all configurations with n partons, $\mathrm{d}\Phi_n$ is the phase space for an n-parton final state with total four-momentum q^μ in $d = 4 - 2\epsilon$ space-time dimensions,

$$\mathrm{d}\Phi_n(p_1, \ldots, p_n; q) = \frac{\mathrm{d}^{d-1}p_1}{2E_1(2\pi)^{d-1}} \cdots \frac{\mathrm{d}^{d-1}p_n}{2E_n(2\pi)^{d-1}} (2\pi)^d \, \delta^d(q - p_1 - \ldots - p_n) \,, \tag{2.2}$$

while S_n is a symmetry factor for identical partons in the final state. The jet function $J_m^{(n)}$ defines the procedure for building m jets out of n partons. The main property of $J_m^{(n)}$ is that the jet observable defined above is collinear and infrared safe as explained in [10, 38]. In general $J_m^{(n)}$ contains θ and δ-functions. $J_m^{(n)}$ can also represent the definition of the n-parton contribution to an event shape observable related to m-jet final states.

$|\mathcal{M}_n|^2$ denotes a squared, colour ordered tree-level n-parton matrix element. Contributions to the squared matrix element which are subleading in the number of colours can equally be treated in the same context, noting that these subleading terms yield configurations where a certain number of essentially non-interacting particles are emitted between a pair of hard radiators. By carrying out the colour algebra, it becomes evident that non-ordered gluon emission inside a colour-ordered system is equivalent to photon emission off the outside legs of the system [37, 36]. For simplicity, these subleading colour contributions are also denoted as squared matrix elements $|\mathcal{M}_m|^2$, although they often correspond purely to interference terms between different amplitudes. The precise definition depends on the number and types of particles involved in the process. However, all colour orderings are summed over in $\sum_m$ with the appropriate colour weighting.

From (2.1), one obtains the leading order approximation to the m-jet cross section by integration over the appropriate phase space.

$$\mathrm{d}\sigma_{\mathrm{LO}} = \int_{\mathrm{d}\Phi_m} \mathrm{d}\sigma^B \,. \tag{2.3}$$

Depending on the jet function used, this cross section can still be differential in certain kinematic quantities.

2.1 NLO antenna subtraction

At NLO, we consider the following m-jet cross section,

$$\mathrm{d}\sigma_{\mathrm{NLO}} = \int_{\mathrm{d}\Phi_{m+1}} \left(\mathrm{d}\sigma^R_{\mathrm{NLO}} - \mathrm{d}\sigma^S_{\mathrm{NLO}}\right) + \left[\int_{\mathrm{d}\Phi_{m+1}} \mathrm{d}\sigma^S_{\mathrm{NLO}} + \int_{\mathrm{d}\Phi_m} \mathrm{d}\sigma^V_{\mathrm{NLO}}\right]. \tag{2.4}$$

The cross section $\mathrm{d}\sigma^R_{\mathrm{NLO}}$ has the same expression as the Born cross section $\mathrm{d}\sigma^B$ (2.1) above except that $n \to m+1$, while $\mathrm{d}\sigma^V_{\mathrm{NLO}}$ is the one-loop virtual correction to the m-parton Born cross section $\mathrm{d}\sigma^B$. The cross section $\mathrm{d}\sigma^S_{\mathrm{NLO}}$ is a (preferably local) counter-term for $\mathrm{d}\sigma^R_{\mathrm{NLO}}$. It has the same unintegrated singular behaviour as $\mathrm{d}\sigma^R_{\mathrm{NLO}}$ in all appropriate limits. Their difference is free of divergences and can be integrated over the $(m+1)$-parton phase space numerically. The subtraction term $\mathrm{d}\sigma^S_{\mathrm{NLO}}$ has to be integrated analytically over all singular regions of the $(m+1)$-parton phase space. The resulting cross section added to the virtual contribution yields an infrared finite result.

Several methods for constructing NLO subtraction terms systematically were proposed in the literature [15, 39−41, 10, 16]. For some of these methods, extension to NNLO was discussed [32, 43, 42] and partly worked out. Up to now, the only method worked out in full detail to NNLO is antenna subtraction [39, 40]. In our calculation of NNLO corrections to three-jet observables, we used this method, which we briefly outline in the following. The details of the method, and a full definition of the notation, can be found in [32].

The basic idea of the antenna subtraction approach is to construct the subtraction terms from antenna functions which encapsulate all singular limits due to the emission of unresolved partons between two colour-connected hard partons. This construction exploits the universal factorisation of both phase space and squared matrix elements in all unresolved limits. The full antenna subtraction term is then constructed by summing products of antenna functions with reduced matrix elements over all possible unresolved configurations.

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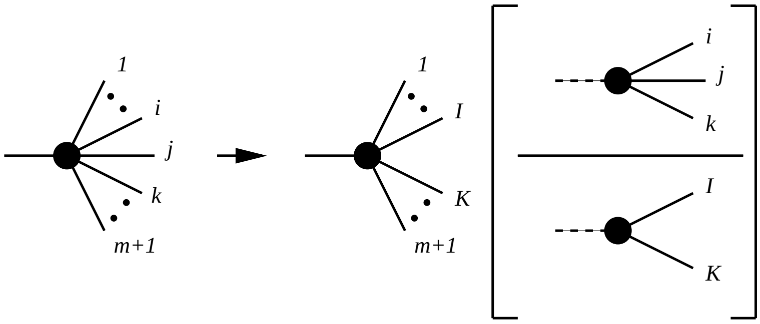

Figure 1: Illustration of NLO antenna factorisation representing the factorisation of both the squared matrix elements and the $(m+1)$-particle phase space. The term in square brackets represents both the antenna function X^0_{ijk} and the antenna phase space $\mathrm{d}\Phi_{X_{ijk}}$.

At NLO, the antenna subtraction term thus reads:

$$\begin{aligned} \mathrm{d}\sigma^S_{\mathrm{NLO}} &= \mathcal{N} \sum_{m+1} \mathrm{d}\Phi_{m+1}(p_1,\ldots,p_{m+1};q) \frac{1}{S_{m+1}} \\ &\times \sum_j X^0_{ijk}\, |\mathcal{M}_m(p_1,\ldots,\tilde{p}_I,\tilde{p}_K,\ldots,p_{m+1})|^2\, J^{(m)}_m(p_1,\ldots,\tilde{p}_I,\tilde{p}_K,\ldots,p_{m+1})\,. \end{aligned} \tag{2.5}$$

The key ingredient is the phase space mapping which relates the original momenta p_i, p_j, p_k describing the two hard radiator partons i and k and the emitted parton j to a redefined on-shell set $\tilde{p}_I, \tilde{p}_K$ which are linear combinations of p_i, p_j, p_k. The phase space mapping yielding this redefinition is described in detail in section 3 below. With this mapping,the phase space factorises,

$$\mathrm{d}\Phi_{m+1}(p_1,\ldots,p_{m+1};q) = \mathrm{d}\Phi_m(p_1,\ldots,\tilde{p}_I,\tilde{p}_K,\ldots,p_{m+1};q)\cdot \mathrm{d}\Phi_{X_{ijk}}(p_i,p_j,p_k;\tilde{p}_I+\tilde{p}_K)\,. \tag{2.6}$$

The other elements of the subtraction term also depend on *either* the original momenta p_i, p_j, p_k *or* the redefined on-shell momenta $\tilde{p}_I, \tilde{p}_K$ but not both. This enables the subtraction term to completely factorise.

To be more specific, both the m-parton amplitude and the jet function $J^{(m)}_m$ depend only on $p_1,\ldots,\tilde{p}_I,\tilde{p}_K,\ldots,p_{m+1}$ i.e. on the redefined on-shell momenta $\tilde{p}_I, \tilde{p}_K$. On the other hand, the tree-level three-parton antenna function X^0_{ijk} depends only on p_i, p_j, p_k. X^0_{ijk} describes all of the configurations (for this colour ordered amplitude) where parton j is unresolved. It can be obtained from appropriately normalised tree-level three-parton squared matrix elements. The antenna factorisation of squared matrix element and phase space can be illustrated pictorially, as displayed in figure 1. Together particles i and k form a colour connected hard antenna that radiates particle j. In doing so, the momenta of the radiators change to form particles I and K. The type of particle may also change.

One can therefore carry out the integration over the antenna phase space appropriate to p_i, p_j and p_k analytically, exploiting the factorisation of the phase space of eq. (2.6). The NLO antenna phase space $\mathrm{d}\Phi_{X_{ijk}}$ is proportional to the three-particle phase space, as can be seen by using $m = 2$ in the above formula. For the analytic integration, we can

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use (2.6) to rewrite each of the subtraction terms in the form,

$$|\mathcal{M}_m|^2\, J_m^{(m)}\, \mathrm{d}\Phi_m \int \mathrm{d}\Phi_{X_{ijk}}\, X_{ijk}^0 = |\mathcal{M}_m|^2\, J_m^{(m)}\, \mathrm{d}\Phi_m\, \mathcal{X}_{ijk}$$

where $|\mathcal{M}_m|^2$, $J_m^{(m)}$ and $\mathrm{d}\Phi_m$ depend only on $p_1, , \ldots, \tilde{p}_I, \tilde{p}_K, \ldots, p_{m+1}$ and $\mathrm{d}\Phi_{X_{ijk}}$ and X_{ijk}^0 depend only on p_i, p_j, p_k. This integration is performed analytically in d dimensions, yielding the integrated three-parton antenna function $\mathcal{X}_{ijk}$, to make the infrared singularities explicit and added directly to the one-loop m-particle contributions.

2.2 NNLO antenna subtraction

At NNLO, the m-jet production is induced by final states containing up to $(m+2)$ partons, including the one-loop virtual corrections to $(m+1)$-parton final states. As at NLO, one has to introduce subtraction terms for the $(m+1)$- and $(m+2)$-parton contributions. Schematically the NNLO m-jet cross section reads,

$$\begin{aligned}
\mathrm{d}\sigma_{\mathrm{NNLO}} = &\int_{\mathrm{d}\Phi_{m+2}} \left(\mathrm{d}\sigma^R_{\mathrm{NNLO}} - \mathrm{d}\sigma^S_{\mathrm{NNLO}}\right) + \int_{\mathrm{d}\Phi_{m+2}} \mathrm{d}\sigma^S_{\mathrm{NNLO}} \\
&+ \int_{\mathrm{d}\Phi_{m+1}} \left(\mathrm{d}\sigma^{V,1}_{\mathrm{NNLO}} - \mathrm{d}\sigma^{VS,1}_{\mathrm{NNLO}}\right) + \int_{\mathrm{d}\Phi_{m+1}} \mathrm{d}\sigma^{VS,1}_{\mathrm{NNLO}} \\
&+ \int_{\mathrm{d}\Phi_m} \mathrm{d}\sigma^{V,2}_{\mathrm{NNLO}}\,, \qquad (2.7)
\end{aligned}$$

where $\mathrm{d}\sigma^S_{\mathrm{NNLO}}$ denotes the real radiation subtraction term coinciding with the $(m+2)$-parton tree level cross section $\mathrm{d}\sigma^R_{\mathrm{NNLO}}$ in all singular limits. Likewise, $\mathrm{d}\sigma^{VS,1}_{\mathrm{NNLO}}$ is the one-loop virtual subtraction term coinciding with the one-loop $(m+1)$-parton cross section $\mathrm{d}\sigma^{V,1}_{\mathrm{NNLO}}$ in all singular limits. Finally, the two-loop correction to the m-parton cross section is denoted by $\mathrm{d}\sigma^{V,2}_{\mathrm{NNLO}}$.

At NNLO, individual antenna functions are obtained from normalised four-parton tree-level and three-parton one-loop matrix elements. The full antenna subtraction term is then constructed by summing products of antenna functions with reduced matrix elements over all possible unresolved configurations.

In $\mathrm{d}\sigma^S_{\mathrm{NNLO}}$, we have to distinguish four different types of unresolved configurations:

(a) One unresolved parton but the experimental observable selects only m jets;

(b) Two colour-connected unresolved partons (colour-connected);

(c) Two unresolved partons that are not colour connected but share a common radiator (almost colour-unconnected);

(d) Two unresolved partons that are well separated from each other in the colour chain (colour-unconnected).

Among those, configuration (a) is properly accounted for by a single tree-level three-parton antenna function like used already at NLO. Configuration (b) requires a tree-level four-parton antenna function (two unresolved partons emitted between a pair of hard partons)

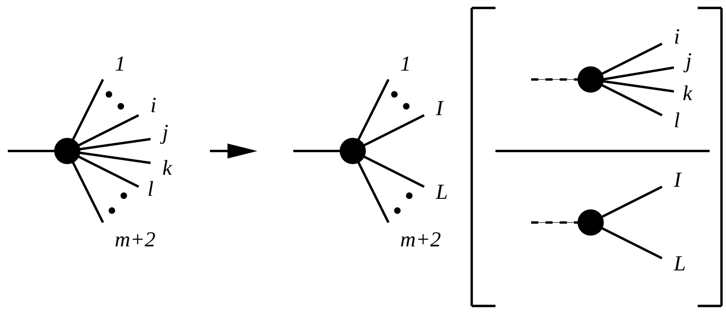

Figure 2: Illustration of NNLO antenna factorisation representing the factorisation of both the squared matrix elements and the $(m+2)$-particle phase space when the unresolved particles are colour connected.

as shown in figure 2, while (c) and (d) are accounted for by products of two tree-level three-parton antenna functions. The subtraction terms for these configurations read:

$$\begin{aligned}
\mathrm{d}\sigma^{S,a}_{\mathrm{NNLO}} &= \mathcal{N} \sum_{m+2} \mathrm{d}\Phi_{m+2}(p_1,\ldots,p_{m+2};q)\frac{1}{S_{m+2}} \\
&\quad \times \Bigg[\sum_{j} X^0_{ijk}\, |\mathcal{M}_{m+1}(p_1,\ldots,\tilde{p}_I,\tilde{p}_K,\ldots,p_{m+2})|^2 \\
&\quad \times J^{(m+1)}_m(p_1,\ldots,\tilde{p}_I,\tilde{p}_K,\ldots,p_{m+2}) \Bigg], \qquad (2.8) \\
\mathrm{d}\sigma^{S,b}_{\mathrm{NNLO}} &= \mathcal{N} \sum_{m+2} \mathrm{d}\Phi_{m+2}(p_1,\ldots,p_{m+2};q)\frac{1}{S_{m+2}} \\
&\quad \times \Bigg[\sum_{jk} \left(X^0_{ijkl} - X^0_{ijk}X^0_{IKl} - X^0_{jkl}X^0_{iJL} \right) \\
&\quad \times |\mathcal{M}_m(p_1,\ldots,\tilde{p}_I,\tilde{p}_L,\ldots,p_{m+2})|^2\, J^{(m)}_m(p_1,\ldots,\tilde{p}_I,\tilde{p}_L,\ldots,p_{m+2}) \Bigg], \qquad (2.9) \\
\mathrm{d}\sigma^{S,c}_{\mathrm{NNLO}} &= -\mathcal{N} \sum_{m+2} \mathrm{d}\Phi_{m+2}(p_1,\ldots,p_{m+2};q)\frac{1}{S_{m+2}} \\
&\quad \times \Bigg[\sum_{j,l} X^0_{ijk}\, x^0_{mlK}\, |\mathcal{M}_m(p_1,\ldots,\tilde{p}_I,\tilde{\tilde{p}}_K,\tilde{\tilde{p}}_M,\ldots,p_{m+2})|^2 \\
&\quad \times J^{(m)}_m(p_1,\ldots,\tilde{p}_I,\tilde{\tilde{p}}_K,\tilde{\tilde{p}}_M,\ldots,p_{m+2}) \\
&\quad + \sum_{j,l} X^0_{klm}\, x^0_{ijK}\, |\mathcal{M}_m(p_1,\ldots,\tilde{\tilde{p}}_I,\tilde{\tilde{p}}_K,\tilde{p}_M,\ldots,p_{m+2})|^2 \\
&\quad \times J^{(m)}_m(p_1,\ldots,\tilde{\tilde{p}}_I,\tilde{\tilde{p}}_K,\tilde{p}_M,\ldots,p_{m+2}) \Bigg], \qquad (2.10)
\end{aligned}$$

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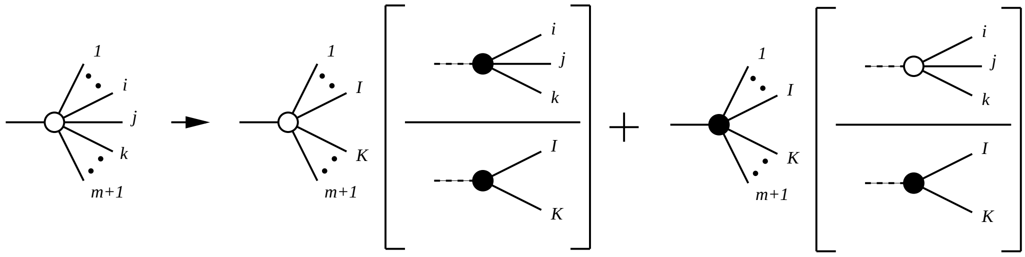

Figure 3: Illustration of NNLO antenna factorisation representing the factorisation of both the one-loop "squared" matrix elements (represented by the white blob) when the unresolved particles are colour connected.

$$
\begin{aligned}
\mathrm{d}\sigma_{\mathrm{NNLO}}^{S,d} = -\mathcal{N} \sum_{m+2} \mathrm{d}\Phi_{m+2}(p_1,\ldots,p_{m+2};q)\frac{1}{S_{m+2}} \\
\times \Bigg[\sum_{j,o} X_{ijk}^0 \, X_{nop}^0 \, |\mathcal{M}_m(p_1,\ldots,\tilde{p}_I,\tilde{p}_K,\ldots,\tilde{p}_N,\tilde{p}_P,\ldots,p_{m+2})|^2 \\
\times J_m^{(m)}(p_1,\ldots,\tilde{p}_I,\tilde{p}_K,\ldots,\tilde{p}_N,\tilde{p}_P,\ldots,p_{m+2}) \Bigg] . \qquad (2.11)
\end{aligned}
$$

Again, the original momenta of the $(m+2)$-parton phase space are denoted by $i,j,\ldots$, while the combined momenta obtained from a phase space mapping are labelled by $I,J,\ldots$. Only the combined momenta appear in the jet function. The phase space mappings appropriate to the different cases are described in detail in section 3 below. X_{ijkl}^0 is a four-parton antenna function, containing all configurations where partons j and k are unresolved, while x_{ijk}^0 is a three-parton sub-antenna function containing only limits where parton j is unresolved with respect to parton i, but not limits where parton j is unresolved with respect to parton k. The factorisation of the phase space is analogous to the factorisation at NLO (2.6), such that integration of these antenna functions over the antenna phase space amounts to inclusive three-particle or four-particle integrals [22].

In single unresolved limits, the one-loop cross section $\mathrm{d}\sigma_{\mathrm{NNLO}}^{V,1}$ is described by the sum of two terms [44]: a tree-level splitting function times a one-loop cross section and a one-loop splitting function times a tree-level cross section. Consequently, the one-loop single unresolved subtraction term $\mathrm{d}\sigma_{\mathrm{NNLO}}^{VS,1}$ is constructed from tree-level and one-loop three-parton antenna functions, as sketched in figure 3. Several other terms in $\mathrm{d}\sigma_{\mathrm{NNLO}}^{VS,1}$ cancel with the results from the integration of terms in the double real radiation subtraction term $\mathrm{d}\sigma_{\mathrm{NNLO}}^{S}$ over the phase space appropriate to one of the unresolved partons, thus ensuring the cancellation of all explicit infrared poles in the difference $\mathrm{d}\sigma_{\mathrm{NNLO}}^{V,1}-\mathrm{d}\sigma_{\mathrm{NNLO}}^{VS,1}$. Explicitly, the one-loop single unresolved subtraction term is given by the sum of the three following contributions:

$$
\begin{aligned}
\mathrm{d}\sigma_{\mathrm{NNLO}}^{VS,1,a} = \mathcal{N} \sum_{m+1} \mathrm{d}\Phi_{m+1}(p_1,\ldots,p_{m+1};q)\frac{1}{S_{m+1}} \\
\times \Bigg[\sum_{ik} -\mathcal{X}_{ijk}^0(s_{ik}) \, |\mathcal{M}_{m+1}(p_1,\ldots,p_i,p_k,\ldots,p_{m+1})|^2
\end{aligned}
$$

$$\times J_m^{(m+1)}(p_1,\ldots,p_i,p_k,\ldots,p_{m+1}) \Bigg], \tag{2.12}$$

$$\begin{aligned} \mathrm{d}\sigma_{\mathrm{NNLO}}^{VS,1,b} &= \mathcal{N} \sum_{m+1} \mathrm{d}\Phi_{m+1}(p_1,\ldots,p_{m+1};q)\frac{1}{S_{m+1}} \\ &\times \sum_j \Bigg[X_{ijk}^0 \,|\mathcal{M}_m^1(p_1,\ldots,\tilde{p}_I,\tilde{p}_K,\ldots,p_{m+1})|^2 \, J_m^{(m)}(p_1,\ldots,\tilde{p}_I,\tilde{p}_K,\ldots,p_{m+1}) \\ &+ X_{ijk}^1 \,|\mathcal{M}_m(p_1,\ldots,\tilde{p}_I,\tilde{p}_K,\ldots,p_{m+1})|^2 \, J_m^{(m)}(p_1,\ldots,\tilde{p}_I,\tilde{p}_K,\ldots,p_{m+1}) \Bigg], \end{aligned} \tag{2.13}$$

$$\begin{aligned} \mathrm{d}\sigma_{\mathrm{NNLO}}^{VS,1,c} &= \mathcal{N} \sum_{m+1} \mathrm{d}\Phi_{m+1}(p_1,\ldots,p_{m+1};q)\frac{1}{S_{m+1}} \\ &\times \Bigg[\sum_{ik} \mathcal{X}_{ijk}^0(s_{ik}) \sum_o X_{nop}^0 \,|\mathcal{M}_m(p_1,\ldots,p_i,p_k,\ldots,\tilde{p}_N,\tilde{p}_P,\ldots,p_{m+1})|^2 \\ &\times J_m^{(m)}(p_1,\ldots,p_i,p_k,\ldots,\tilde{p}_N,\tilde{p}_P,\ldots,p_{m+1}) \Bigg], \end{aligned} \tag{2.14}$$

In here, X_{ijk}^1 denotes a one-loop three-parton antenna function.

Finally, all remaining terms in $\mathrm{d}\sigma_{\mathrm{NNLO}}^S$ and $\mathrm{d}\sigma_{\mathrm{NNLO}}^{VS,1}$ have to be integrated over the four-parton and three-parton antenna phase spaces. After integration, the infrared poles are rendered explicit and cancel with the infrared pole terms in the two-loop squared matrix element $\mathrm{d}\sigma_{\mathrm{NNLO}}^{V,2}$.

The subtraction terms $\mathrm{d}\sigma_{\mathrm{NLO}}^S$, $\mathrm{d}\sigma_{\mathrm{NNLO}}^S$ and $\mathrm{d}\sigma_{\mathrm{NNLO}}^{VS,1}$ require three different types of antenna functions corresponding to the different pairs of hard partons forming the antenna: quark-antiquark, quark-gluon and gluon-gluon antenna functions. In the past [39, 40], NLO antenna functions were constructed by imposing definite properties in all single unresolved limits (two collinear limits and one soft limit for each antenna). This procedure turns out to be impractical at NNLO, where each antenna function must have definite behaviours in a large number of single and double unresolved limits. Instead, we derived these antenna functions in a systematic manner from physical matrix elements known to possess the correct limits. The quark-antiquark antenna functions can be obtained directly from the $e^+e^- \to 2j$ real radiation corrections at NLO and NNLO [29]. For quark-gluon and gluon-gluon antenna functions, effective Lagrangians [45, 46] are used to obtain tree-level processes yielding a quark-gluon or gluon-gluon final state. The antenna functions are then derived from the real radiation corrections to these processes. Quark-gluon antenna functions were derived [47] from the purely QCD (i.e. non-supersymmetric) NLO and NNLO corrections to the decay of a heavy neutralino into a massless gluino plus partons [45], while gluon-gluon antenna functions [48] result from the QCD corrections to Higgs boson decay into partons [46].

All tree-level three-parton and four-parton antenna functions and three-parton one-loop antenna functions are listed in [32]. Their integration over the antenna phase space amounts to performing inclusive infrared-divergent three-parton and four-parton phase space integrals. Techniques for evaluating these integrals are described in [22, 49], and all integrated antenna functions are documented in [32].

3. Phase space mappings

The subtraction terms for single and double unresolved configurations described in the previous section involve the mapping of the momenta appearing in the antenna functions into combined momenta, which appear in the reduced matrix elements and in the jet function.

At NLO, one needs momentum mappings from three partons to two partons,

$$\mathcal{F}^{(3\to 2)}: \{p_i, p_j, p_k\} \to \{\tilde{p}_I, \tilde{p}_K\}.$$

At NNLO, several further mappings are needed. First and foremost, one needs momentum mappings from four partons to two partons,

$$\mathcal{F}^{(4\to 2)}: \{p_i, p_j, p_k, p_l\} \to \{\tilde{p}_I, \tilde{p}_L\}.$$

In addition, repeated mappings from three partons to two partons are also required.

For the subtraction and phase space factorisation to work correctly, all mappings must satisfy the following requirements (specified here for the example of $\mathcal{F}^{(4\to 2)}$):

1. momentum conservation: $\tilde{p}_I + \tilde{p}_L = p_i + p_j + p_k + p_l$.
2. the new momenta should be on-shell: $\tilde{p}_I^2 = 0$, $\tilde{p}_L^2 = 0$.
3. the new momenta should reduce to the appropriate original momenta in the exact singular limits, e.g. $\tilde{p}_I = p_i + p_j + p_k$, $\tilde{p}_L = p_l$ in the (i,j,k) triple collinear limit.
4. the mapping should not introduce spurious singularities.

The momentum mappings we use follow largely those worked out in [40, 43]. The different types of mappings needed for our calculation are described in detail in the following subsections.

3.1 Mapping for single unresolved configurations

In the single unresolved limit where parton j becomes unresolved and i,k are the hard radiators, the momenta of the partons i,j,k are mapped to $\tilde{p}_I = \widetilde{(ij)}$ and $\tilde{p}_K = \widetilde{(kj)}$ in the following way:

$$\begin{aligned}
\tilde{p}_I &= x\, p_i + r\, p_j + z\, p_k \\
\tilde{p}_K &= (1-x)\, p_i + (1-r)\, p_j + (1-z)\, p_k \;,
\end{aligned}$$

where

$$\begin{aligned}
x &= \frac{1}{2(s_{ij}+s_{ik})}\Big[(1+\rho)\, s_{ijk} - 2\, r\, s_{jk}\Big] \\
z &= \frac{1}{2(s_{jk}+s_{ik})}\Big[(1-\rho)\, s_{ijk} - 2\, r\, s_{ij}\Big] \\
\rho^2 &= 1 + \frac{4\, r(1-r)\, s_{ij} s_{jk}}{s_{ijk} s_{ik}} \;.
\end{aligned} \tag{3.1}$$

The parameter r can be chosen conveniently, we use [40]

$$r = \frac{s_{jk}}{s_{ij} + s_{jk}} \; .$$

3.2 Mappings from five partons to three partons

In the construction of the subtraction terms for the five-parton channel, one encounters, both the four-parton antenna functions (double unresolved limits), and products of two three-parton antenna functions, which compensate for the single unresolved limits of the double unresolved subtraction terms. The former require a $4 \to 2$ mapping, while the latter require, in general, a $5 \to 3$ mapping affecting all momenta. Both types of mappings are described in the following.

3.2.1 Double unresolved configurations

In the double unresolved limit where partons i_2 and i_3 become unresolved and i_1, i_4 are the hard radiators, the partons $i_1, \ldots, i_4$ are mapped to the partons j_1, j_2 with momenta

$$\tilde{p}_{j_1} \equiv \widetilde{(i_1 i_2 i_3)} \; , \; \tilde{p}_{j_2} \equiv \widetilde{(i_4 i_3 i_2)} \; . \tag{3.2}$$

We will use the shorthand notation (3.2) extensively in the following, keeping in mind however that $\tilde{p}_{j_1}$ and $\tilde{p}_{j_2}$ are linear combinations of all four original momenta with a mapping $\mathcal{F}^{(4\to 2)} : \{p_{i_1}, p_{i_2}, p_{i_3}, p_{i_4}\} \to \{\tilde{p}_{j_1}, \tilde{p}_{j_2}\}$ given by:

$$\begin{aligned} \tilde{p}_{j_1} &= x \, p_{i_1} + r_1 \, p_{i_2} + r_2 \, p_{i_3} + z \, p_{i_4} \\ \tilde{p}_{j_2} &= (1-x) \, p_{i_1} + (1-r_1) \, p_{i_2} + (1-r_2) \, p_{i_3} + (1-z) \, p_{i_4} \; . \end{aligned} \tag{3.3}$$

Defining $s_{kl} = (p_{i_k} + p_{i_l})^2$, the coefficients are given by [43]

$$\begin{aligned}
r_1 &= \frac{s_{23} + s_{24}}{s_{12} + s_{23} + s_{24}} \\
r_2 &= \frac{s_{34}}{s_{13} + s_{23} + s_{34}} \\
x &= \frac{1}{2(s_{12} + s_{13} + s_{14})} \Bigg[(1+\rho) \, s_{1234} - r_1 \, (s_{23} + 2 \, s_{24}) - r_2 \, (s_{23} + 2 \, s_{34}) \\
&\quad + (r_1 - r_2) \frac{s_{12} s_{34} - s_{13} s_{24}}{s_{14}} \Bigg] \\
z &= \frac{1}{2(s_{14} + s_{24} + s_{34})} \Bigg[(1-\rho) \, s_{1234} - r_1 \, (s_{23} + 2 \, s_{12}) - r_2 \, (s_{23} + 2 \, s_{13}) \\
&\quad - (r_1 - r_2) \frac{s_{12} s_{34} - s_{13} s_{24}}{s_{14}} \Bigg] \\
\rho &= \Bigg[1 + \frac{(r_1 - r_2)^2}{s_{14}^2 \, s_{1234}^2} \, \lambda(s_{12} \, s_{34}, s_{14} \, s_{23}, s_{13} \, s_{24}) \\
&\quad + \frac{1}{s_{14} \, s_{1234}} \Bigg\{ 2 \, (r_1 \, (1 - r_2) + r_2 (1 - r_1))(s_{12} s_{34} + s_{13} s_{24} - s_{23} s_{14}) \\
&\quad + 4 \, r_1 \, (1 - r_1) \, s_{12} s_{24} + 4 \, r_2 \, (1 - r_2) \, s_{13} s_{34} \Bigg\} \Bigg]^{\frac{1}{2}} \; , \\
\lambda(u, v, w) &= u^2 + v^2 + w^2 - 2(uv + uw + vw) \; .
\end{aligned}$$

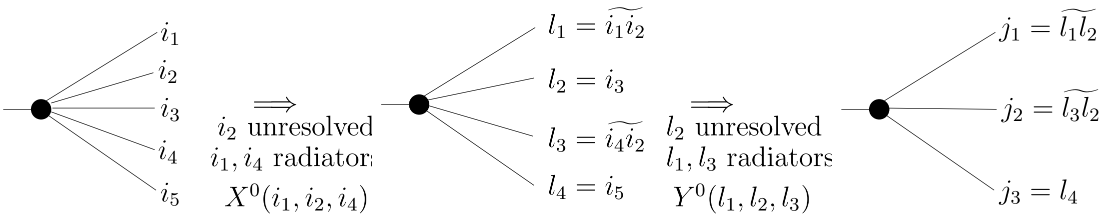

Figure 4: $\mathcal{F}_{\mathrm{B}}^{(5\to 4\to 3)}(i_1, i_2, i_3, i_4, i_5; j_1, j_2, j_3)$: both combined partons (l_1 and l_3) are radiators in the second step, original parton i_3 becomes unresolved.

3.2.2 Iteration of single unresolved configurations

The four-particle antennae X^0_{ijkl} contain by construction all colour-connected double unresolved limits of the $(m+2)$-parton matrix element where partons j and k become unresolved. However, X^0_{ijkl} can also become singular in single unresolved limits, where it does not coincide with limits of the matrix element. These limits have to be subtracted as indicated in eq. 2.9 and as described in section 2.3 of [32]. In the limit where j becomes unresolved between i and k, the four-particle antenna collapses to the three-particle antenna $X^0(i, j, k)$ and a three-particle "remainder matrix element" $X^0(I, K, l)$, which also has the form of a three-particle antenna. If one of the partons I, K, l becomes unresolved in a second single emission, the resulting momenta in these limits coincide with those from a double unresolved configuration as defined in subsection 3.2.1. Therefore we need momentum mappings corresponding to such a "two-step" emission in order to be able to subtract the spurious singularities of the four-particle antennae X^0_{ijkl}.

As explained in section 2.2 above and in [32], we have to distinguish between colour-connected unresolved partons and almost colour-unconnected unresolved partons. If the unresolved partons are colour-connected, we use the momentum mappings $\mathcal{F}_{\mathrm{B,C}}^{(5\to 4\to 3)}$: $\{p_{i_1}, p_{i_2}, p_{i_3}, p_{i_4}, p_{i_5}\} \to \{p_{l_1}, p_{l_2}, p_{l_3}, p_{l_4}\} \to \{\tilde{p}_{j_1}, \tilde{p}_{j_2}, \tilde{p}_{j_3}\}$, where the different types B,C are described in more detail below. These two-step mappings first map the four partons $i_1, \ldots, i_4$ making up the four-particle antenna to three "intermediate" partons l_1, l_2, l_3, and then map the three intermediate partons to two partons j_1, j_2. The fifth parton $i_5 = l_4 = j_3$ only acts as a spectator. An additional type of mapping, denoted by $\mathcal{F}_{\mathrm{K}}^{(5\to 4\to 3)}$, is needed for subtraction terms of two almost colour-unconnected unresolved partons, as defined in eq. 2.10 and in [32], and involves redefinitions of all five initial partons. All three mappings are depicted in figures 4–6.

In the first step, one of the partons $\{i_1, i_2, i_3, i_4\}$ becomes unresolved. The momentum $i_5 = l_4$ is always unaffected by the mapping. The precise definition of the resulting momenta l_1, l_2, l_3 depends on the mapping. In the first step of the two-step mapping $\mathcal{F}_{\mathrm{B}}^{(5\to 4\to 3)}$ we

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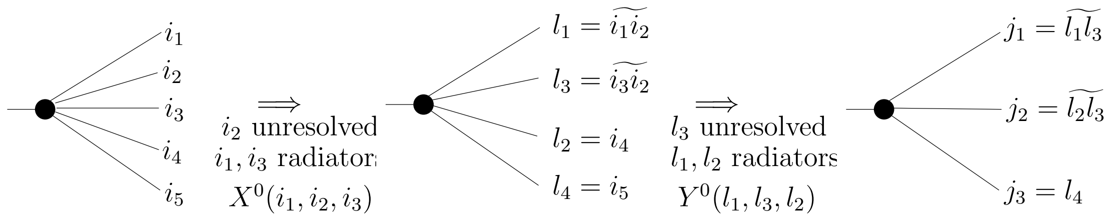

Figure 5: $\mathcal{F}_{\rm C}^{(5\to 4\to 3)}(i_1, i_2, i_3, i_4, i_5; j_1, j_2, j_3)$: one of the combined partons (here l_3) becomes unresolved between l_1 and l_2 in the second step.

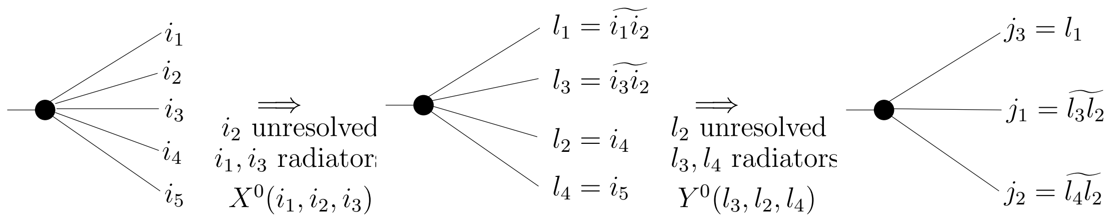

Figure 6: $\mathcal{F}_{\rm K}^{(5\to 4\to 3)}(i_1, i_2, i_3, i_4, i_5; j_1, j_2, j_3)$: i_3 is the shared hard radiator, $i_4 = l_2$ becomes unresolved between l_3 and l_4 in the second step.

have

$$
\begin{aligned}
l_1 &= x_1\, i_1 + r_1\, i_2 + z_1\, i_4 \equiv \widetilde{i_1 i_2} \\
l_2 &= i_3 \\
l_3 &= (1-x_1)\, i_1 + (1-r_1)\, i_2 + (1-z_1)\, i_4 \equiv \widetilde{i_4 i_2} \\
l_4 &= i_5\ ,
\end{aligned}
\tag{3.4}
$$

while for the mappings $\mathcal{F}_{\rm C}^{(5\to 4\to 3)}$ and $\mathcal{F}_{\rm K}^{(5\to 4\to 3)}$, the first step is of the form

$$
\begin{aligned}
l_1 &= x_1\, i_1 + r_1\, i_2 + z_1\, i_3 \equiv \widetilde{i_1 i_2} \\
l_2 &= i_4 \\
l_3 &= (1-x_1)\, i_1 + (1-r_1)\, i_2 + (1-z_1)\, i_3 \equiv \widetilde{i_3 i_2} \\
l_4 &= i_5\ .
\end{aligned}
\tag{3.5}
$$

The momentum i_2 is always the unresolved one, denoted generically by i_u in the following. The two hard radiators denoted by i_a and i_b depend on the mapping. In $\mathcal{F}_{\rm B}^{(5\to 4\to 3)}$, i_1 and i_4 are the hard radiators, while in $\mathcal{F}_{\rm C}^{(5\to 4\to 3)}$ and $\mathcal{F}_{\rm K}^{(5\to 4\to 3)}$, i_1 and i_3 are the hard radiators.

Note that mappings of type C apply to all configurations where one of the combined partons becomes unresolved in the second step, so they also include the cases where the roles of l_1 and l_3 are interchanged.

	First step			Second step		
Mapping type	i_u	i_a, i_b	i_s	l_u	l_a, l_b	l_s
B	i_2	i_1, i_4	i_3	l_2	l_1, l_3	l_4
C	i_2	i_1, i_3	i_4	l_3 or l_1	(l_1 or l_3), l_2	l_4
K	i_2	i_1, i_3	i_4	l_2	l_3, l_4	l_1

Table 1: Identification of the unresolved, radiator and spectator momenta for both steps of the momentum mappings $\mathcal{F}^{(5\to4\to3)}_{\mathrm{B,C,K}}$.

The coefficients r_1, x_1, z_1 appearing in eqs. (3.4) and (3.5) are given by eq. (3.1),

$$\begin{aligned} r_1 &= \frac{s_{ub}}{s_{au} + s_{ub}} \\ x_1 &= \frac{1}{2(s_{au} + s_{ab})}\Big[(1+\rho)\, s_{aub} - 2\, r_1\, s_{ub}\Big] \\ z_1 &= \frac{1}{2(s_{ub} + s_{ab})}\Big[(1-\rho)\, s_{aub} - 2\, r_1\, s_{au}\Big] \\ \rho &= \left[1 + \frac{4\, r_1(1-r_1)\, s_{au}s_{ub}}{s_{ab}s_{aub}}\right]^{\frac{1}{2}} \end{aligned} \tag{3.6}$$

where now $s_{ub} = (i_u + i_b)^2$ etc.

In the second step, one of the intermediate partons $\{l_1, l_2, l_3\}$ becomes unresolved. The resulting momenta j_1, j_2, j_3 are defined as

$$j_1 = x_2\, l_a + r_2\, l_u + z_2\, l_b \equiv \widetilde{l_a l_u} \tag{3.7}$$

$$j_2 = (1-x_2)\, l_a + (1-r_2)\, l_u + (1-z_2)\, l_b \equiv \widetilde{l_b l_u} \tag{3.8}$$

$$j_3 = l_r\ ,$$

where again l_u denotes the momentum which is unresolved in the second step, l_a, l_b are the radiators and l_r does not take part in the second recombination step. The coefficients r_2, x_2, z_2 are defined analogously to eq. (3.6), where now $s_{ub} = (l_u + l_b)^2$ etc.

The combination of antenna functions associated with the repeated unresolved singularity is thus

$$X^0(i_a, i_u, i_b)Y^0(l_a, l_u, l_b) \tag{3.9}$$

as indicated in figures 4–6 where X^0 and Y^0 generically stand for three-particle tree antenna functions. The identification of the radiator and unresolved momenta $i_a, \ldots, l_b$ for the different mappings $\mathcal{F}^{(5\to4\to3)}_{\mathrm{B,C,K}}$ can be read off from table 1 and is also illustrated in figures 4–6.

3.3 Decomposition of antenna functions into sub-antennae

The antenna phase space mappings require two uniquely identified hard radiator momenta and an ordered emission of the unresolved partons.

With the antenna functions of [32], it is not always possible to uniquely identify the hard momenta, especially if more than one final state parton is a gluon. Moreover, in the four-parton antenna functions (containing two unresolved partons) at subleading colour, the emission is not colour-ordered. It was already outlined in [32] that in both these cases, a further decomposition of the antenna functions into different sub-antennae configurations is required.

The tree-level three-parton antenna functions $D_3^0(1_q, 3_g, 4_g)$ (quark-gluon-gluon) and $F_3^0(1_g, 2_g, 3_g)$ (gluon-gluon-gluon) contain more than one antenna configuration, since each gluon can become unresolved. Their decomposition was discussed in [32], it reads:

$$D_3^0(1,3,4) = d_3^0(1,3,4) + d_3^0(1,4,3) \,, \tag{3.10}$$
$$F_3^0(1,2,3) = f_3^0(1,3,2) + f_3^0(3,2,1) + f_3^0(2,1,3) \,, \tag{3.11}$$

with

$$d_3^0(1,3,4) = \frac{1}{s_{134}^2}\left(\frac{2s_{134}^2 s_{14}}{s_{13}s_{34}} + \frac{s_{14}s_{34}+s_{34}^2}{s_{13}} + \frac{s_{13}s_{14}}{s_{34}} + \frac{5}{2}s_{134} + \frac{1}{2}s_{34}\right) + \mathcal{O}(\epsilon) \,, \tag{3.12}$$
$$f_3^0(1,3,2) = \frac{1}{s_{123}^2}\left(2\frac{s_{123}^2 s_{12}}{s_{13}s_{23}} + \frac{s_{12}s_{13}}{s_{23}} + \frac{s_{12}s_{23}}{s_{13}} + \frac{8}{3}s_{123}\right) + \mathcal{O}(\epsilon) \,. \tag{3.13}$$

With this decomposition, the sub-antennae $d_3^0(i,j,k)$ and $f_3^0(i,j,k)$ contain only a soft singularity associated with gluon j, and collinear singularities $i \parallel j$ and $k \parallel j$, such that i and k can be identified as hard radiators. Soft singularities associated with i or k and the collinear singularity $i \parallel k$, which were present in the full antenna functions, are now contained in different sub-antennae, obtained by permutations of the momenta. Therefore, each sub-antenna can have a unique phase space mapping $(i,j,k) \to (\widetilde{ij}, \widetilde{kj})$.

The one-loop three-parton antenna functions $D_3^1(1_q, 3_g, 4_g)$ (quark-gluon-gluon) and $F_3^1(1_g, 2_g, 3_g)$ (gluon-gluon-gluon) can be decomposed according to the same pattern, exploiting the fact that each can be written as a function proportional to its tree-level counterpart plus a function which is not singular in any unresolved limit.

The decomposition of tree-level four-parton antenna functions is more involved, especially since both single and double unresolved limits have to be accounted for properly. It turns out to be very useful to introduce the following combinations of three-parton antenna functions:

$$\begin{aligned}
Q_3^0(1,3,2) &= d_3^0(1,3,2) - A_3^0(1,3,2) \,, \\
R_3^0(1,3,2) &= Q_3^0(1,3,2) - Q_3^0(1,2,3) \\
S_3^0(1,3,2) &= Q_3^0(1,3,2) + Q_3^0(1,2,3) + E_3^0(1,3,2) \,.
\end{aligned} \tag{3.14}$$

None of these contains any soft limit or collinear $1 \parallel i$ limit. Only Q_3^0 and R_3^0 contain a $2 \parallel 3$ limit, while S_3^0 is also finite in this limit, owing to the $\mathcal{N}=1$ supersymmetry relation between the tree-level splitting functions,

$$P_{q\bar{q}\to G}(z) + P_{gg\to G}(z) = P_{qg\to Q}(z) + P_{qg\to Q}(1-z) \,. \tag{3.15}$$

Among the tree-level four-parton antenna functions, only $\tilde{A}_4^0(1_q, 3_g, 4_g, 2_{\bar{q}})$ (quark-gluon-gluon-antiquark at subleading colour), $D_4^0(1_q, 3_g, 4_g, 5_g)$ (quark-gluon-gluon-gluon),

$E_4^0(1_q, 3_{q'}, 4_{\bar{q}'}, 5_g)$ (quark-quark-antiquark-gluon at leading colour), $F_4^0(1_g, 2_g, 3_g, 4_g)$ (gluon-gluon-gluon-gluon) as well as $G_4^0(1_g, 3_q, 4_{\bar{q}}, 2_g)$ and $\tilde{G}_4^0(1_g, 3_q, 4_{\bar{q}}, 2_g)$ (gluon-quark-antiquark-gluon at leading and subleading colour) must be decomposed into sub-antennae. In the context of the three-jet calculation discussed here, F_4^0, G_4^0 and $\tilde{G}_4^0$ do not contribute and will not be discussed further.

The decomposition of $\tilde{A}_4^0(1_q, 3_g, 4_g, 2_{\bar{q}})$ is needed because both gluons can become collinear either with quark 1_q or with antiquark $2_{\bar{q}}$. The two possible phase space mappings are of the $\mathcal{F}^{(5\to3)}$ type described in section 3.2.1

$$\text{(a): } (1,3,4,2) \to (\widetilde{134}, \widetilde{243}) \qquad \text{(b): } (1,4,3,2) \to (\widetilde{143}, \widetilde{234})$$

each allow only gluons adjacent to quark or antiquark to become collinear. To disentangle the different sub-antennae, it is sufficient to partial-fraction the antenna function in the different collinear denominators, as done in [32]. With these, we decompose

$$\tilde{A}_4^0(1,3,4,2) = \tilde{A}_{4,a}^0(1,3,4,2) + \tilde{A}_{4,b}^0(1,3,4,2), \tag{3.16}$$

with

$$\begin{aligned} \tilde{A}_{4,a}^0(1,3,4,2) &= \tilde{a}_4^0(1,3,4,2) + \tilde{a}_4^0(2,4,3,1) , \\ \tilde{A}_{4,b}^0(1,3,4,2) &= \tilde{A}_{4,a}^0(1,4,3,2) , \end{aligned} \tag{3.17}$$

where $\tilde{a}_4^0(i,j,k,l)$ contains only singularities for $i \parallel j$ or $k \parallel l$ and was defined in [32]. With this decomposition, $\tilde{A}_{4,a}^0(1,3,4,2)$ contains only $1 \parallel 3$ and $2 \parallel 4$ singularities, and can be used with the $(1,3,4,2) \to (\widetilde{134}, \widetilde{243})$ mapping. Since this decomposition is straightforward, we refrain from spelling it out explicitly in the subtraction terms presented in subsequent sections.

The leading-colour quark-quark-antiquark-gluon antenna function $E_4^0(1_q, 3_{q'}, 4_{\bar{q}'}, 5_g)$ contains limits where either the quark-antiquark pair $(3_{q'}, 4_{\bar{q}'})$ or the gluon 5_g can become soft. Since these limits yield different hard radiator partons, they can not be accounted for in a single phase space mapping, but require two separate $\mathcal{F}^{(5\to3)}$ mappings:

$$\text{(a): } (1,3,4,5) \to (\widetilde{134}, \widetilde{543}) , \qquad \text{(b): } (1,5,4,3) \to (\widetilde{154}, \widetilde{345}) .$$

By analysing the different triple and single collinear limits of E_4^0, one finds the following decomposition:

$$E_{4,a}^0(1,3,4,5) = B_4^0(1,3,4,5) + E_3^0(5,4,3)\, Q_3^0(1, \widetilde{(34)}, \widetilde{(54)}) , \tag{3.18}$$

$$E_{4,b}^0(1,3,4,5) = E_4^0(1,3,4,5) - E_{4,a}^0(1,3,4,5) . \tag{3.19}$$

After this decomposition, $E_{4,a}^0$ can be used with mapping (a) and $E_{4,b}^0$ with mapping (b). The above decomposition also ensures a well-defined behaviour in all double and single unresolved limits (see [32] for a definition of the splitting factors):

$$E_{4,a}^0(1,3,4,5) \stackrel{3_{q'}\to 0, 4_{\bar{q}'}\to 0}{\longrightarrow} S_{15}(3,4) ,$$

$$
\begin{aligned}
E^0_{4,b}(1,3,4,5) &\xrightarrow{3_{q'}\|4_{\bar{q}'},5_g\to 0} S_{1;543}(z)\,\frac{1}{s_{34}}\,P_{q\bar{q}\to G}(z)\,,\\
E^0_{4,a}(1,3,4,5) &\xrightarrow{1_q\|3_{q'}\|4_{\bar{q}'}} P^{\rm non-ident.}_{134\to Q}(w,x,y)\,,\\
E^0_{4,a}(1,3,4,5)+E^0_{4,b}(1,3,4,5) &\xrightarrow{3_{q'}\|4_{\bar{q}'}\|5_g} P_{543\to G}(w,x,y)\,,\\
E^0_{4,b}(1,3,4,5) &\xrightarrow{1_q\|5_g,3_{q'}\|4_{\bar{q}'}} \frac{1}{s_{34}s_{15}}\,P_{qg\to Q}(z)\,P_{q\bar{q}\to G}(y)\,,
\end{aligned}
\tag{3.20}
$$

$$
\begin{aligned}
E^0_{4,b}(1,3,4,5) &\xrightarrow{5_g\to 0} S_{154}\;E^0_3(1,3,4)\,,\\
E^0_{4,a}(1,3,4,5) &\xrightarrow{3_{q'}\|4_{\bar{q}'}} \frac{1}{s_{34}}\,P_{q\bar{q}\to G}(z)\;d^0_3(1,(34),5)+\text{ang.}\,,\\
E^0_{4,b}(1,3,4,5) &\xrightarrow{3_{q'}\|4_{\bar{q}'}} \frac{1}{s_{34}}\,P_{q\bar{q}\to G}(z)\;d^0_3(1,5,(34))+\text{ang.}\,,\\
E^0_{4,b}(1,3,4,5) &\xrightarrow{1_q\|5_g} \frac{1}{s_{15}}\,P_{qg\to Q}(z)\;E^0_3((15),3,4)\,,\\
E^0_{4,b}(1,3,4,5) &\xrightarrow{4_{\bar{q}'}\|5_g} \frac{1}{s_{45}}\,P_{qg\to Q}(z)\;E^0_3(1,3,(45))\,,
\end{aligned}
\tag{3.21}
$$

while all other limits are zero. It can be seen that only the triple collinear $3 \parallel 4 \parallel 5$ and the single collinear $3 \parallel 4$ limits receive contributions from both phase space mappings. This is unavoidable, since both these limits match onto the double soft $(3_{q'}, 4_{\bar{q}'}) \to 0$ and the soft 5_g limits, which belong to different mappings.

The decomposition of the quark-gluon-gluon-gluon antenna function $D^0_4(1_q, 3_g, 4_g, 5_g)$, is more involved, since any pair of two gluons can become soft. We consider four different $\mathcal{F}^{(5\to 3)}$ mappings:

$$
\begin{aligned}
&\text{(a): } (1,3,4,5) \to (\widetilde{134},\widetilde{543})\,, \qquad &&\text{(b): } (1,5,4,3) \to (\widetilde{154},\widetilde{345})\,,\\
&\text{(c): } (1,3,5,4) \to (\widetilde{135},\widetilde{453})\,, \qquad &&\text{(d): } (1,5,3,4) \to (\widetilde{153},\widetilde{435})\,.
\end{aligned}
$$

The numerous different double and single unresolved limits of this antenna function can be disentangled very elegantly by repeatedly exploiting the $\mathcal{N}=1$ supersymmetry relation [37] among the different triple collinear splitting functions [37, 35, 50]. Using this relation, one can show that the following left-over combination is finite in all single unresolved and double unresolved limits:

$$
\begin{aligned}
D^0_{4,l}(1,3,4,5) = D^0_4(1,3,4,5) - \Bigg[& A^0_4(1,3,4,5) + A^0_4(1,5,4,3)\\
& -\frac{1}{2}\left(\tilde{E}^0_4(1,3,5,4) + \tilde{E}^0_4(1,5,3,4)\right) + \tilde{A}^0_4(1,3,5,4)\\
& -E^0_4(1,5,4,3) + B^0_4(1,5,4,3) + C^0_4(1,4,5,3)\\
& -E^0_4(1,3,4,5) + B^0_4(1,3,4,5) + C^0_4(1,4,3,5)\\
& +A^0_3(1,3,4)\,S^0_3(\widetilde{(13)},\widetilde{(43)},5) + A^0_3(1,5,4)\,S^0_3(\widetilde{(15)},\widetilde{(45)},3)\\
& +A^0_3(3,4,5)\,S^0_3(1,\widetilde{(54)},\widetilde{(34)})\Bigg]\,.
\end{aligned}
\tag{3.22}
$$

Starting from the terms in this expression, the following sub-antennae can be constructed:

$$\begin{aligned}
D_{4,a}^0(1,3,4,5) &= \frac{1}{2} D_{4,l}^0(1,3,4,5) + A_4^0(1,3,4,5) - \frac{1}{2}\tilde{E}_4^0(1,3,5,4) \\
&\quad + A_3^0(1,3,4)\, S_3^0(\widetilde{(13)},\widetilde{(43)},5) \\
&\quad + \frac{1}{2} A_3^0(3,4,5)\, \Big(S_3^0(1,\widetilde{(54)},\widetilde{(34)}) - R_3^0(1,\widetilde{(54)},\widetilde{(34)}) \Big) \\
&\quad - E_3^0(5,4,3) Q_3^0(1,\widetilde{(54)},\widetilde{(34)}) - A_3^0(1,3,4)\, E_3^0(\widetilde{(13)},\widetilde{(43)},5) \\
&\quad - A_3^0(1,3,4)\, Q_3^0(\widetilde{(13)},5,\widetilde{(43)})\ , \\
D_{4,b}^0(1,3,4,5) &= D_{4,a}^0(1,5,4,3)\ , \\
D_{4,c}^0(1,3,4,5) &= \tilde{A}_{4,a}^0(1,3,5,4) - E_4^0(1,5,4,3) + B_4^0(1,5,4,3) + C_4^0(1,4,5,3) \\
&\quad + E_3^0(5,4,3) Q_3^0(1,\widetilde{(54)},\widetilde{(34)}) + A_3^0(1,3,4)\, E_3^0(\widetilde{(13)},\widetilde{(43)},5) \\
&\quad + a_3^0(1,3,4)\, Q_3^0(\widetilde{(13)},5,\widetilde{(43)}) + a_3^0(4,5,1)\, Q_3^0(\widetilde{(15)},3,\widetilde{(45)})\ , \\
D_{4,d}^0(1,3,4,5) &= D_{4,c}^0(1,5,4,3)\ .
\end{aligned} \tag{3.23}$$

With this decomposition, each $D_{4,i}^0$ contains only singularities appropriate to phase space mapping (i). The sum of the $D_{4,i}^0$ adds to D_4^0:

$$D_{4,a}^0 + D_{4,b}^0 + D_{4,c}^0 + D_{4,d}^0 = D_4^0\ , \tag{3.24}$$

such that only D_4^0 must be integrated analytically over the antenna phase space.

The above decomposition disentangles the different double and single unresolved limits:

$$\begin{aligned}
D_{4,a}^0(1,3,4,5) &\xrightarrow{3_g\to 0,4_g\to 0} S_{1345}\ , \\
D_{4,b}^0(1,3,4,5) &\xrightarrow{4_g\to 0,5_g\to 0} S_{1543}\ , \\
D_{4,c}^0(1,3,4,5) + D_{4,d}^0(1,3,4,5) &\xrightarrow{3_g\to 0,5_g\to 0} S_{134}\, S_{154}\ , \\
D_{4,c}^0(1,3,4,5) + D_{4,d}^0(1,3,4,5) &\xrightarrow{1_q\|5_g,3_g\to 0} S_{4;315}(z)\, \frac{1}{s_{15}}\, P_{qg\to Q}(1-z)\ , \\
D_{4,a}^0(1,3,4,5)+D_{4,c}^0(1,3,4,5)+D_{4,d}^0(1,3,4,5) &\xrightarrow{4_g\|5_g,3_g\to 0} S_{1;345}(z) \frac{1}{s_{45}}\, P_{gg\to G}(z)\ , \\
D_{4,a}^0(1,3,4,5) &\xrightarrow{1_q\|3_g,4_g\to 0} S_{5;431}(z)\, \frac{1}{s_{13}}\, P_{qg\to Q}(z)\ , \\
D_{4,b}^0(1,3,4,5) &\xrightarrow{1_q\|5_g,4_g\to 0} S_{3;451}(z)\, \frac{1}{s_{15}}\, P_{qg\to Q}(z)\ , \\
D_{4,c}^0(1,3,4,5) + D_{4,d}^0(1,3,4,5) &\xrightarrow{1_q\|3_g,5_g\to 0} S_{4;513}(z)\, \frac{1}{s_{13}}\, P_{qg\to Q}(1-z)\ , \\
D_{4,b}^0(1,3,4,5)+D_{4,c}^0(1,3,4,5)+D_{4,d}^0(1,3,4,5) &\xrightarrow{3_g\|4_g,5_g\to 0} S_{1;543}(z)\, \frac{1}{s_{34}}\, P_{gg\to G}(z)\ , \\
D_{4,a}^0(1,3,4,5) &\xrightarrow{1_q\|3_g\|4_g} P_{134\to Q}(w,x,y)\ , \\
D_{4,b}^0(1,3,4,5) &\xrightarrow{1_q\|5_g\|4_g} P_{154\to Q}(w,x,y)\ , \\
D_{4,c}^0(1,3,4,5) + D_{4,d}^0(1,3,4,5) &\xrightarrow{1_q\|3_g\|5_g} \tilde{P}_{135\to Q}(w,x,y)\ , \\
D_4^0(1,3,4,5) &\xrightarrow{3_g\|4_g\|5_g} P_{345\to G}(w,x,y)\ ,
\end{aligned}$$

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$$
\begin{aligned}
D^0_{4,a}(1,3,4,5)+D^0_{4,c}(1,3,4,5) &\xrightarrow{1_q\|3_g,4_g\|5_g} \frac{1}{s_{13}s_{45}}\,P_{qg\to Q}(z)\;P_{gg\to G}(y)\;,\\
D^0_{4,b}(1,3,4,5)+D^0_{4,d}(1,3,4,5) &\xrightarrow{1_q\|5_g,3_g\|4_g} \frac{1}{s_{15}s_{34}}\,P_{qg\to Q}(z)\;P_{gg\to G}(y)\;,\\
D^0_{4,a}(1,3,4,5) &\xrightarrow{3_g\to 0} S_{134}\;d^0_3(1,4,5)\;,\\
D^0_{4,c}(1,3,4,5)+D^0_{4,d}(1,3,4,5) &\xrightarrow{3_g\to 0} S_{134}\;d^0_3(1,5,4)\;,\\
D^0_{4,a}(1,3,4,5) &\xrightarrow{4_g\to 0} S_{345}\;d^0_3(1,3,5)\;,\\
D^0_{4,b}(1,3,4,5) &\xrightarrow{4_g\to 0} S_{345}\;d^0_3(1,5,3)\;,\\
D^0_{4,b}(1,3,4,5) &\xrightarrow{5_g\to 0} S_{154}\;d^0_3(1,4,3)\;,\\
D^0_{4,c}(1,3,4,5)+D^0_{4,d}(1,3,4,5) &\xrightarrow{5_g\to 0} S_{154}\;d^0_3(1,3,4)\;,\\
D^0_{4,a}(1,3,4,5) &\xrightarrow{1_q\|3_g} \frac{1}{s_{13}}\;P_{qg\to Q}(z)\;d^0_3((13),4,5)\;,\\
D^0_{4,c}(1,3,4,5) &\xrightarrow{1_q\|3_g} \frac{1}{s_{13}}\;P_{qg\to Q}(z)\;d^0_3((13),5,4)\;,\\
D^0_{4,b}(1,3,4,5) &\xrightarrow{1_q\|5_g} \frac{1}{s_{15}}P_{qg\to Q}(z)d^0_3((15),4,3),\\
D^0_{4,d}(1,3,4,5) &\xrightarrow{1_q\|5_g} \frac{1}{s_{15}}P_{qg\to Q}(z)d^0_3((15),3,4),\\
D^0_{4,a}(1,3,4,5) &\xrightarrow{3_g\|4_g} \frac{1}{s_{34}}P_{gg\to G}(z)d^0_3(1,(34),5)+\text{ang.},\\
D^0_{4,b}(1,3,4,5)+D^0_{4,d}(1,3,4,5) &\xrightarrow{3_g\|4_g} \frac{1}{s_{34}}P_{gg\to G}(z)d^0_3(1,5,(34))+\text{ang.},\\
D^0_{4,b}(1,3,4,5) &\xrightarrow{4_g\|5_g} \frac{1}{s_{45}}P_{gg\to G}(z)d^0_3(1,(45),3)+\text{ang.},\\
D^0_{4,a}(1,3,4,5)+D^0_{4,c}(1,3,4,5) &\xrightarrow{4_g\|5_g} \frac{1}{s_{45}}P_{gg\to G}(z)d^0_3(1,3,(45))+\text{ang.}.
\end{aligned}
\tag{3.25}
$$

All other limits are vanishing. It can be seen that certain limits are shared among several antenna functions, which can be largely understood due to two reasons:

1. in a gluon-gluon collinear splitting, either gluon can become soft, and the gluon-gluon splitting function is always shared between two sub-antennae, as in (3.10), (3.11) to disentangle the two soft limits.

2. the unresolved emission of gluons 3_g and 5_g is shared between the mappings (c) and (d) according to the decomposition of the non-ordered antenna function $\tilde{A}^0_4$, which distributes the soft limit of either gluon between both mappings.

3.4 Angular terms

The angular terms in the single unresolved limits are associated with a gluon splitting into two gluons or into a quark-antiquark pair. They average to zero after integration over the antenna phase space. To ensure numerical stability and reliability, this average has to take place within each phase space mapping. We have checked this to be the case for the above decompositions of E_4^0 and D_4^0. The angular average in single collinear limits can be made using the standard momentum parametrisation [10, 51] for the $i \parallel j$ limit:

$$
\begin{aligned}
p_i^\mu &= z p^\mu + k_\perp^\mu - \frac{k_\perp^2}{z}\frac{n^\mu}{2p\cdot n}\,, \qquad & p_j^\mu &= (1-z)p^\mu - k_\perp^\mu - \frac{k_\perp^2}{1-z}\frac{n^\mu}{2p\cdot n}\,,\\
\text{with } 2p_i\cdot p_j &= -\frac{k_\perp^2}{z(1-z)}\,, \qquad & p^2 &= n^2 = 0\,.
\end{aligned}
$$

In this p^μ denotes the collinear momentum direction, and n^μ is an auxiliary vector. The collinear limit is approached by $k_\perp^2 \to 0$.

In the simple collinear $i \parallel j$ limit of the four-parton antenna functions $X_4^0(i,j,k,l)$, one chooses $n = p_k$ to be one of the non-collinear momenta, such that the antenna function can be expressed in terms of p, n, $k_\perp$ and p_l. Expanding in $k_\perp^\mu$ yields only non-vanishing scalar products of the form $p_l \cdot k_\perp$. Expressing the integral over the antenna phase space in the (p,n) centre-of-mass frame, the angular average can be carried out as

$$
\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi\,(p_l\cdot k_\perp) = 0\,, \qquad \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi\,(p_l\cdot k_\perp)^2 = -k_\perp^2\,\frac{p\cdot p_l\, n\cdot p_l}{p\cdot n}\,. \tag{3.26}
$$

Higher powers of $k_\perp^\mu$ are not sufficiently singular to contribute to the collinear limit. Using the above average, we could analytically verify the cancellation of angular terms within each single phase space mapping, which is independent on the choice of the reference vector n_μ.

An angular average is also required for the cancellation of $1/\epsilon$-poles in the four-parton one-loop subtraction terms in the N^2 and N^0 colour factors. In either of these colour factors, one finds that the difference $\mathrm{d}\sigma^{V,1}_{\mathrm{NNLO}} - \mathrm{d}\sigma^{VS,1}_{\mathrm{NNLO}}$ contains left-over poles of the form

$$
\frac{1}{\epsilon^2} X_3^0(1,i,2)\, Y_3^0(\widetilde{1i},j,\widetilde{2i}) J_3^{(3)}(\widetilde{1i},j,\widetilde{2i}) \left\{ s_{\widetilde{1i}j}^{-\epsilon} + s_{\widetilde{2i}j}^{-\epsilon} - s_{1j}^{-\epsilon} - s_{2j}^{-\epsilon} - s_{1i2}^{-\epsilon} + s_{12}^{-\epsilon} \right\}\,, \tag{3.27}
$$

where X_3^0 and Y_3^0 are tree-level three-parton antenna functions. Expanding this in ϵ, one finds that only the $1/\epsilon^2$ poles cancel, while the $1/\epsilon$-pole is as follows:

$$
X_3^0(1,i,2)\, Y_3^0(\widetilde{1i},j,\widetilde{2i}) J_3^{(3)}(\widetilde{1i},j,\widetilde{2i})\, \frac{1}{\epsilon} \left(\log \frac{s_{1j}s_{2j}}{s_{12}} - \log \frac{s_{\widetilde{1i}j}s_{\widetilde{2i}j}}{s_{1i2}} \right)\,. \tag{3.28}
$$

The finite ϵ^0-contribution from (3.28) is also non-vanishing, and has precisely the form required to ensure proper subtraction of all soft and collinear limits in $\mathrm{d}\sigma^{V,1}_{\mathrm{NNLO}} - \mathrm{d}\sigma^{VS,1}_{\mathrm{NNLO}}$.

To ensure full cancellation of the $1/\epsilon$-poles, one has to show that (3.28) vanishes already after integration over the antenna phase space of $(1,i,2)$, since all other momenta appear in the jet definition. The second term in the bracket integrates just to a constant, since it does not depend on $(1,i,2)$, while the first term is a lot more tricky because it combines integration momenta $(1,2)$ with the fixed momentum j.

To carry out this integration, we work in a frame where the z-axis is defined by the momenta $(\widetilde{1i})$ and $(\widetilde{2i})$. The momentum p_j can be chosen in the (x,z)-plane. We fix p_1 to be along the direction of $p_{\widetilde{1i}}$, the angle between p_1 and p_2 (p_j) is called Θ_2 (Θ_j), and

LO	$\gamma^* \to q\,\bar{q}g$	tree level
NLO	$\gamma^* \to q\,\bar{q}g$	one loop
	$\gamma^* \to q\,\bar{q}\,gg$	tree level
	$\gamma^* \to q\,\bar{q}\,q\bar{q}$	tree level
NNLO	$\gamma^* \to q\,\bar{q}g$	two loop
	$\gamma^* \to q\,\bar{q}\,gg$	one loop
	$\gamma^* \to q\,\bar{q}\,q\,\bar{q}$	one loop
	$\gamma^* \to q\,\bar{q}\,q\,\bar{q}\,g$	tree level
	$\gamma^* \to q\,\bar{q}\,g\,g\,g$	tree level
	$\gamma^* \to ggg$	(one loop)^2

Table 2: The partonic channels contributing to $e^+e^- \to 3$ jets.

the polar angle of p_2 is ϕ_2. The antenna functions do not depend on the orientation of the $(1,i,2)$ phase space in the $(\widetilde{1i})$, $(\widetilde{2i})$ frame, and are therefore independent of ϕ_2. If $\Theta_2 \geq \Theta_j$, we boost along the z-direction to a frame where $p_{j,z}=0$, while for $\Theta_2 < \Theta_j$, we boost to a frame where $p_{2,z}=0$. This boost leaves the second term in (3.28) invariant:

$$\log \frac{s_{\widetilde{1i}j}s_{\widetilde{2i}j}}{s_{1i2}} = \log E_{j,T}^2 = \log E_{j,T}^{\prime 2}, \tag{3.29}$$

whereas the first term becomes either $(\Theta_2 \geq \Theta_j)$:

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi_2' \log \frac{s_{1j}s_{2j}}{s_{12}} &= \log E_{j,T}^{\prime 2} + \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi_2' \log \left(\frac{2(1-\sin\Theta_2'\cos\phi_2')}{1-\cos\Theta_2'} \right) \\ &= \log E_{j,T}^{\prime 2} \,, \end{aligned} \tag{3.30}$$

or $(\Theta_2 < \Theta_j)$:

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi_2' \log \frac{s_{1j}s_{2j}}{s_{12}} &= \log E_{j,T}^{\prime 2} + \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi_2' \log \left(2\left(1-\cos\Theta_j'\right)\left(1-\cos\phi_2'\sin\Theta_j'\right)\right) \\ &= \log E_{j,T}^{\prime 2} \,, \end{aligned} \tag{3.31}$$

such that in either case the sum of both terms in (3.28) exactly cancels after averaging over ϕ_2'.

4. Parton-level contributions to $e^+e^- \to 3$ jets up to NNLO

Three-jet production at tree-level is induced by the decay of a virtual photon (or other neutral gauge boson) into a quark-antiquark-gluon final state. At higher orders, this process receives corrections from extra real or virtual particles. The individual partonic channels that contribute through to NNLO are shown in table 2.

According to the structure of the coupling to the external vector boson, one distinguishes non-singlet and singlet contributions. The non-singlet contributions arise from the interference of amplitudes where the external gauge boson couples to the same quark lines, while the pure singlet contribution is due to the interference of amplitudes where the external gauge boson couples to different quark lines. Up to NLO, only non-singlet contributions appear. It is only at NNLO, that the first non-vanishing singlet terms are allowed. These appear in the tree-level $\gamma^* \to q\,\bar{q}\,q\,\bar{q}\,g$ process, the one-loop $\gamma^* \to q\,\bar{q}\,gg$ and $\gamma^* \to q\,\bar{q}\,q\,\bar{q}$ processes and the two-loop $\gamma^* \to q\,\bar{q}g$ process. All these processes yield both non-singlet and singlet contributions. The $\gamma^* \to ggg$ process, which is mediated by a closed quark loop, is entirely a singlet contribution. In four-jet observables at $\mathcal{O}(\alpha_s^3)$, the singlet contributions were found to be extremely small [52]. Also, the singlet contribution from three-gluon final states to three-jet observables was found to be negligible [53].

Matrix elements and subtraction terms at NLO and NNLO can be naturally decomposed according to their colour structure. The cross section at NLO receives contributions from three different colour factors:

$$\mathrm{d}\sigma_{\mathrm{NLO}} = \mathrm{d}\sigma_{\mathrm{NLO,N}} + \mathrm{d}\sigma_{\mathrm{NLO,1/N}} + \mathrm{d}\sigma_{\mathrm{NLO,N_F}} \,. \tag{4.1}$$

The NNLO contribution to the cross section receives contributions from seven different colour factors:

$$\begin{aligned}\mathrm{d}\sigma_{\mathrm{NNLO}} =\;& \mathrm{d}\sigma_{\mathrm{NNLO},N^2} + \mathrm{d}\sigma_{\mathrm{NNLO},N^0} + \mathrm{d}\sigma_{\mathrm{NNLO},1/N^2} \\ &+\mathrm{d}\sigma_{\mathrm{NNLO},N_F N} + \mathrm{d}\sigma_{\mathrm{NNLO},N_F/N} + \mathrm{d}\sigma_{\mathrm{NNLO},N_F^2} + \mathrm{d}\sigma_{\mathrm{NNLO},N_{F,\gamma}} \,. \end{aligned} \tag{4.2}$$

The first six terms in this equation are non-singlet contributions, the last term is the numerically unimportant singlet contribution.

In the following, we list the matrix elements for the contributing partonic channels shown in table 2 and discuss their structure.

4.1 Tree-level matrix elements for up to five partons

The tree-level amplitude $M^0_{q\bar{q}(n-2)g}$ for a virtual photon to produce a quark-antiquark pair and $(n-2)$-gluons,

$$\gamma^*(q) \to q(p_1)\bar{q}(p_2)g(p_3)\ldots g(p_n)$$

can be expressed as sum over the permutations of the colour ordered amplitude $\mathcal{M}^0_{A,n}$ of the possible orderings for the gluon colour indices

$$M^0_{q\bar{q}(n-2)g} = ie(\sqrt{2}g)^{n-2} \sum_{(i,\ldots,k)\in P(3,\ldots,n)} (T^{a_i}\cdots T^{a_n})_{i_1 i_2}\, \mathcal{M}^0_{A,n}(p_1,p_3,\ldots,p_n,p_2) \,. \tag{4.3}$$

The squared matrix elements for $n = 3,\ldots,5$, summed over gluon polarisations, but excluding symmetry factors for identical particles, are given by,

$$\left|M^0_{q\bar{q}g}\right|^2 = N_3\, A^0_3(1_q,3_g,2_{\bar{q}}) \,, \tag{4.4}$$

$$\left|M^0_{q\bar{q}gg}\right|^2 = N_4 \left[\sum_{(i,j)\in P(3,4)} N A^0_4(1_q,i_g,j_g,2_{\bar{q}}) - \frac{1}{N}\tilde{A}^0_4(1_q,3_g,4_g,2_{\bar{q}})\right] \,, \tag{4.5}$$

$$
\left|M^0_{q\bar{q}ggg}\right|^2 = N_5 \left[\sum_{(i,j,k)\in P(3,4,5)} \left(N^2 A^0_5(1_q, i_g, j_g, k_g, 2_{\bar{q}}) - \tilde{A}^0_5(1_q, i_g, j_g, k_g, 2_{\bar{q}}) \right) \right.
$$

$$
\left. + \left(\frac{N^2+1}{N^2} \right) \bar{A}^0_5(1_q, 3_g, 4_g, 5_g, 2_{\bar{q}}) \right] , \tag{4.6}
$$

where,

$$
N_n = 4\pi\alpha \sum_q e_q^2 \left(g^2\right)^{(n-2)} \left(N^2-1\right) \left|\mathcal{M}^0_{q\bar{q}}\right|^2 , \tag{4.7}
$$

and

$$
\left|\mathcal{M}^0_{q\bar{q}}\right|^2 = 4(1-\epsilon)q^2. \tag{4.8}
$$

The squared colour-ordered matrix elements A^0_3, A^0_4 and $\tilde{A}^0_4$ are given in [32]. For the five parton case [54],

$$
A^0_5(1_q, i_g, j_g, k_g, 2_{\bar{q}}) \left|\mathcal{M}^0_{q\bar{q}}\right|^2 = \left| \mathcal{M}^0_{A,5}(p_1, p_i, p_j, p_k, p_2) \right|^2 \tag{4.9}
$$

$$
\tilde{A}^0_5(1_q, i_g, j_g, k_g, 2_{\bar{q}}) \left|\mathcal{M}^0_{q\bar{q}}\right|^2 = \left| \mathcal{M}^0_{A,5}(p_1, p_i, p_j, p_k, p_2) \right. \tag{4.10}
$$

$$
\left. + \mathcal{M}^0_{A,5}(p_1, p_i, p_k, p_j, p_2) + \mathcal{M}^0_{A,5}(p_1, p_k, p_i, p_j, p_2) \right|^2 ,
$$

$$
\bar{A}^0_5(1_q, i_g, j_g, k_g, 2_{\bar{q}}) \left|\mathcal{M}^0_{q\bar{q}}\right|^2 = \left| \sum_{(i,j,k)\in P(3,\ldots,5)} \mathcal{M}^0_{A,5}(p_1, p_i, p_j, p_k, p_2) \right|^2 . \tag{4.11}
$$

In the subleading colour contribution $\tilde{A}^0_5$, gluon k is effectively photon-like, while in the subsubleading colour contribution (also called Abelian contribution), $\bar{A}^0_5$, all three gluons are effectively photon-like. Photon-like gluons do not couple to three- and four-gluon vertices, and there are no simple collinear limits as any two photon-like gluons become collinear. As a consequence, the only colour-connected pair in $\bar{A}^0_5$ are the quark and antiquark.

The tree-level amplitude for

$$
\gamma^*(q) \to q(p_1)\bar{q}(p_2)q'(p_3)\bar{q}'(p_4)
$$

is given by

$$
M^0_{q\bar{q}q'\bar{q}'} = ie_1 g^2 \delta_{q_1 q_2}\delta_{q_3 q_4} \left(\delta_{i_1 i_4}\delta_{i_3 i_2} - \frac{1}{N}\delta_{i_1 i_2}\delta_{i_3 i_4} \right) \mathcal{M}^0_{B,4}(p_1, p_2, p_3, p_4)
$$

$$
+ (1 \leftrightarrow 3, 2 \leftrightarrow 4) , \tag{4.12}
$$

where $\delta_{q_1 q_2}\delta_{q_3 q_4}$ indicates the quark flavours. The amplitude $\mathcal{M}^0_{B,4}(p_1, p_2, p_3, p_4)$ thus denotes the contribution from the $q_1\bar{q}_2$-pair coupling to the vector boson. The identical quark amplitude is obtained

$$
M^0_{q\bar{q}q\bar{q}} = M^0_{q\bar{q}q'\bar{q}'} - (2 \leftrightarrow 4). \tag{4.13}
$$

The resulting four-quark squared matrix elements, summed over final state quark flavours and including symmetry factors are given by

$$\begin{aligned}
\left|M^0_{4q}\right|^2 &= \sum_{q,q'} \left|M_{q\bar{q}q'\bar{q}'}\right|^2 + \sum_q \left|M_{q\bar{q}q\bar{q}}\right|^2 \\
&= N_4 \Bigg[N_F B^0_4(1_q, 3_q, 4_{\bar{q}}, 2_{\bar{q}}) - \frac{1}{N} \left(C^0_4(1_q, 3_q, 4_{\bar{q}}, 2_{\bar{q}}) + C^0_4(2_{\bar{q}}, 4_{\bar{q}}, 3_q, 1_q) \right) \\
&\qquad + N_{F,\gamma}\, \hat{B}^0_4(1_q, 3_q, 4_{\bar{q}}, 2_{\bar{q}}) \Bigg] , \qquad (4.14)
\end{aligned}$$

where

$$\begin{aligned}
B^0_4(1_q, 3_q, 4_{\bar{q}}, 2_{\bar{q}}) \left|\mathcal{M}^0_{q\bar{q}}\right|^2 &= \left|\mathcal{M}^0_{B,4}(p_1, p_2, p_3, p_4)\right|^2 , \\
C^0_4(1_q, 3_q, 4_{\bar{q}}, 2_{\bar{q}}) \left|\mathcal{M}^0_{q\bar{q}}\right|^2 &= -\mathrm{Re}\Big(\mathcal{M}^0_{B,4}(p_1, p_2, p_3, p_4) \mathcal{M}^{0,\dagger}_{B,4}(p_1, p_4, p_3, p_2) \Big) , \qquad (4.15) \\
\hat{B}^0_4(1_q, 3_q, 4_{\bar{q}}, 2_{\bar{q}}) \left|\mathcal{M}^0_{q\bar{q}}\right|^2 &= \mathrm{Re}\Big(\mathcal{M}^0_{B,4}(p_1, p_2, p_3, p_4) \mathcal{M}^{0,\dagger}_{B,4}(p_3, p_4, p_1, p_2) \Big) . \qquad (4.16)
\end{aligned}$$

Explicit expressions for B^0_4 and C^0_4 are given in [32]. The last term, $\hat{B}^0_4$, is proportional to the charge weighted sum of the quark flavours, $N_{F,\gamma}$, which for electromagnetic interactions is given by,

$$N_{F,\gamma} = \frac{\left(\sum_q e_q\right)^2}{\sum_q e_q^2} . \qquad (4.17)$$

It is relevant only for observables where the final state quark charge can be determined.

There are four colour structures in the tree-level amplitude for

$$\gamma^*(q) \to q(p_1)\bar{q}(p_2)q'(p_3)\bar{q}'(p_4)g(p_5)$$

which reads

$$\begin{aligned}
M_{q\bar{q}q'\bar{q}'g} &= i e_1 g^3 \sqrt{2} \delta_{q_1 q_2} \delta_{q_3 q_4} \\
&\quad \times \Bigg[T^{a_5}_{i_1 i_4} \delta_{i_3 i_2} \mathcal{M}^{0,a}_{B,5}(p_1, p_2, p_3, p_4, p_5) - \frac{1}{N} T^{a_5}_{i_1 i_2} \delta_{i_3 i_4} \mathcal{M}^{0,c}_{B,5}(p_1, p_2, p_3, p_4, p_5) \\
&\quad + T^{a_5}_{i_3 i_2} \delta_{i_1 i_4} \mathcal{M}^{0,b}_{B,5}(p_1, p_2, p_3, p_4, p_5) - \frac{1}{N} T^{a_5}_{i_3 i_4} \delta_{i_1 i_2} \mathcal{M}^{0,d}_{B,5}(p_1, p_2, p_3, p_4, p_5) \Bigg] \\
&\quad + (1 \leftrightarrow 3, 2 \leftrightarrow 4) . \qquad (4.18)
\end{aligned}$$

The amplitude $\mathcal{M}^{0,x}_{B,5}(p_1, p_2, p_3, p_4, p_5)$ for $x = a, \ldots, d$ denotes the contribution from the $q_1\bar{q}_2$-pair coupling to the vector boson. Due to the colour decomposition, the following relation holds between the leading and subleading colour amplitudes:

$$\begin{aligned}
\mathcal{M}^{0,e}_{B,5}(p_1, p_2, p_3, p_4, p_5) &= \mathcal{M}^{0,a}_{B,5}(p_1, p_2, p_3, p_4, p_5) + \mathcal{M}^{0,b}_{B,5}(p_1, p_2, p_3, p_4, p_5) \\
&= \mathcal{M}^{0,c}_{B,5}(p_1, p_2, p_3, p_4, p_5) + \mathcal{M}^{0,d}_{B,5}(p_1, p_2, p_3, p_4, p_5) . \qquad (4.19)
\end{aligned}$$

As before, the identical quark matrix element is obtained by permuting the antiquark momenta,

$$M^0_{q\bar{q}q\bar{q}g} = M^0_{q\bar{q}q'\bar{q}'g} - (2 \leftrightarrow 4). \tag{4.20}$$

The squared matrix element, summed over flavours and including symmetry factors is given by,

$$\begin{aligned}
\left|M^0_{4qg}\right|^2 &= \sum_{q,q'} \left|M_{q\bar{q}q'\bar{q}'g}\right|^2 + \sum_{q} \left|M_{q\bar{q}q\bar{q}g}\right|^2 \qquad (4.21)\\
&= N_5 \Bigg[N N_F \left(B_5^{0,a}(1_q, 5_g, 4_{\bar{q}'}; 3_{q'}, 2_{\bar{q}}) + B_5^{0,b}(1_q, 4_{\bar{q}'}; 3_{q'}, 5_g, 2_{\bar{q}}) \right) \\
&\quad + \frac{N_F}{N} \left(B_5^{0,c}(1_q, 5_g, 2_{\bar{q}}; 3_{q'}, 4_{\bar{q}'}) + B_5^{0,d}(1_q, 2_{\bar{q}}; 3_{q'}, 5_g, 4_{\bar{q}'}) - 2B_5^{0,e}(1_q, 2_{\bar{q}}; 3_{q'}, 4_{\bar{q}'}; 5_g) \right) \\
&\quad - C_5^0(1_q, 3_q, 4_{\bar{q}}, 5_g, 2_{\bar{q}}) + \left(\frac{N^2+1}{N^2} \right) \left(\tilde{C}_5^0(1_q, 3_q, 4_{\bar{q}}, 5_g, 2_{\bar{q}}) + \tilde{C}_5^0(2_{\bar{q}}, 4_{\bar{q}}, 3_q, 5_g, 1_q) \right) \\
&\quad - N N_{F,\gamma} \left(\hat{B}_5^{0,a}(1_q, 5_g, 4_{\bar{q}'}; 3_{q'}, 2_{\bar{q}}) + \hat{B}_5^{0,b}(1_q, 4_{\bar{q}'}; 3_{q'}, 5_g, 2_{\bar{q}}) - \hat{B}_5^{0,e}(1_q, 4_{\bar{q}'}; 3_{q'}, 2_{\bar{q}}, 5_g) \right) \\
&\quad + \frac{N_{F,\gamma}}{N} \left(\hat{B}_5^{0,c}(1_q, 5_g, 2_{\bar{q}}; 3_{q'}, 4_{\bar{q}'}) + \hat{B}_5^{0,d}(1_q, 2_{\bar{q}}; 3_{q'}, 5_g, 4_{\bar{q}'}) + \hat{B}_5^{0,e}(1_q, 2_{\bar{q}}; 3_{q'}, 4_{\bar{q}'}; 5_g) \right) \Bigg],
\end{aligned}$$

where for $x = a, \ldots, e$

$$B_5^{0,x}(\ldots) \left|\mathcal{M}^0_{q\bar{q}}\right|^2 = |\mathcal{M}^{0,x}_{B,5}(p_1, p_2, p_3, p_4, p_5)|^2, \tag{4.22}$$

$$\hat{B}_5^{0,x}(\ldots) \left|\mathcal{M}^0_{q\bar{q}}\right|^2 = \mathrm{Re}\left(\mathcal{M}^{0,x}_{B,5}(p_1, p_2, p_3, p_4, p_5) \mathcal{M}^{0,x,\dagger}_{B,5}(p_3, p_4, p_1, p_2, p_5) \right), \tag{4.23}$$

and

$$\begin{aligned}
C_5^0(1_q, 3_q, 4_{\bar{q}}, 5_g, 2_{\bar{q}}) \left|M^0_{q\bar{q}}\right|^2 = -2\mathrm{Re}\Big(&\mathcal{M}^{0,a}_{B,5}(p_1, p_2, p_3, p_4, p_5) \mathcal{M}^{0,c,\dagger}_{B,5}(p_1, p_4, p_3, p_2, p_5) \\
&+ \mathcal{M}^{0,b}_{B,5}(p_1, p_2, p_3, p_4, p_5) \mathcal{M}^{0,d,\dagger}_{B,5}(p_1, p_4, p_3, p_2, p_5) \\
&+ \mathcal{M}^{0,a}_{B,5}(p_1, p_2, p_3, p_4, p_5) \mathcal{M}^{0,d,\dagger}_{B,5}(p_3, p_2, p_1, p_4, p_5) \\
&+ \mathcal{M}^{0,b}_{B,5}(p_1, p_2, p_3, p_4, p_5) \mathcal{M}^{0,c,\dagger}_{B,5}(p_3, p_2, p_1, p_4, p_5) \Big), \qquad (4.24)
\end{aligned}$$

$$\tilde{C}_5^0(1_q, 3_q, 4_{\bar{q}}, 5_g, 2_{\bar{q}}) \left|M^0_{q\bar{q}}\right|^2 = -\mathrm{Re}\left(\mathcal{M}^{0,e}_{B,5}(p_1, p_2, p_3, p_4, p_5) \mathcal{M}^{0,e,\dagger}_{B,5}(p_1, p_4, p_3, p_2, p_5) \right). \tag{4.25}$$

4.2 One-loop matrix elements for up to four partons

The renormalised one-loop amplitude $M^1_{q\bar{q}g}$ for a virtual photon to produce a quark-antiquark pair together with a single gluon,

$$\gamma^*(q) \to q(p_1)\bar{q}(p_2)g(p_3)$$

contains a single colour structure such that

$$M^1_{q\bar{q}g} = ie\sqrt{2}g \left(\frac{g^2}{16\pi^2} \right) T^{a_3}_{i_1 i_2} \mathcal{M}^1_{A,3}(p_1, p_3, p_2)\,. \tag{4.26}$$

Unless stated otherwise, the renormalisation scale is set to $\mu^2 = q^2$.

The interference of the one-loop amplitude with the three-parton tree-level amplitude (4.3) is given by

$$2\mathrm{Re}\left(M^{0,\dagger}_{q\bar{q}g} M^1_{q\bar{q}g}\right) = N_3 \left(\frac{\alpha_s}{2\pi}\right) A_3^{(1\times 0)}(1_q, 3_g, 2_{\bar{q}}) \,, \tag{4.27}$$

where

$$\begin{aligned} A_3^{(1\times 0)}(1_q, 3_g, 2_{\bar{q}}) = & \Big(N \left[A_3^1(1_q, 3_g, 2_{\bar{q}}) + \mathcal{A}_2^1(s_{123}) A_3^0(1_q, 3_g, 2_{\bar{q}})\right] \\ & -\frac{1}{N}\left[\tilde{A}_3^1(1_q, 3_g, 2_{\bar{q}}) + \mathcal{A}_2^1(s_{123}) A_3^0(1_q, 3_g, 2_{\bar{q}})\right] + N_F \hat{A}_3^1(1_q, 3_g, 2_{\bar{q}})\Big) \,, \end{aligned} \tag{4.28}$$

where A_3^1, $\tilde{A}_3^1$ and $\hat{A}_3^1$ are given up to $\mathcal{O}(\epsilon^0)$ in [32].

Moreover, the one-loop process $\gamma^* \to ggg$ also yields three-jet final states. Since this process has no tree-level counterpart, it does only contribute at NNLO. Its amplitude can be denoted as [53]

$$M^1_{ggg} = i \sum_q e_q \sqrt{2} g \left(\frac{g^2}{16\pi^2}\right) d^{a_1 a_2 a_3} \mathcal{M}^1_{C,3}(p_1, p_2, p_3) \,. \tag{4.29}$$

The one-loop corrections to $\gamma^* \to 4$ partons have been available for some time [21]. The one-loop amplitude for

$$\gamma^*(q) \to q(p_1)\bar{q}(p_2)g(p_3)g(p_4)$$

contains two colour structures,

$$\begin{aligned} M^1_{q\bar{q}gg} = & \, ie2g^2 \left(\frac{g^2}{16\pi^2}\right) \\ & \times \Bigg[\sum_{(i,j)\in P(3,4)} (T^{a_i} T^{a_j})_{i_1 i_2} \Big(N \mathcal{M}^{1,a}_{A,4}(p_1, p_i, p_j, p_2) - \frac{1}{N} \mathcal{M}^{1,b}_{A,4}(p_1, p_i, p_j, p_2) \\ & + N_F \mathcal{M}^{1,c}_{A,4}(p_1, p_i, p_j, p_2) + \frac{\sum_q e_q}{e} \mathcal{M}^{1,e}_{A,4}(p_1, p_i, p_j, p_2)\Big) \\ & + \frac{1}{2}\delta^{a_i a_j} \delta_{i_1 i_2} \mathcal{M}^{1,d}_{A,4}(p_1, p_3, p_4, p_2) \Bigg] \,, \end{aligned} \tag{4.30}$$

where

$$\mathcal{M}^{1,d}_{A,4}(p_1, p_3, p_4, p_2) = \mathcal{M}^{1,d}_{A,4}(p_1, p_4, p_3, p_2). \tag{4.31}$$

The "squared" matrix element is the interference between the tree-level and one-loop amplitudes,

$$\begin{aligned}
2\left|M^{0,\dagger}_{q\bar{q}gg}M^{1}_{q\bar{q}gg}\right| &= N_4\left(\frac{\alpha_s}{2\pi}\right)\Bigg[\sum_{(i,j)\in P(3,4)}\Bigg(N^2 A_4^{1,a}(1_q,i_g,j_g,2_{\bar{q}})\\
&\quad -A_4^{1,b}(1_q,i_g,j_g,2_{\bar{q}})+NN_F A_4^{1,c}(1_q,i_g,j_g,2_{\bar{q}})+NN_{F,\gamma}\,A_4^{1,e}(1_q,i_g,j_g,2_{\bar{q}})\Bigg)\\
&\quad -\Bigg(\tilde{A}_4^{1,a}(1_q,3_g,4_g,2_{\bar{q}})-\tilde{A}_4^{1,d}(1_q,3_g,4_g,2_{\bar{q}})-\frac{1}{N^2}\tilde{A}_4^{1,b}(1_q,3_g,4_g,2_{\bar{q}})\\
&\quad +\frac{N_F}{N}\tilde{A}_4^{1,c}(1_q,3_g,4_g,2_{\bar{q}})+\frac{N_{F,\gamma}}{N}\tilde{A}_4^{1,e}(1_q,3_g,4_g,2_{\bar{q}})\Bigg)\Bigg], \qquad (4.32)
\end{aligned}$$

where for $x=a,\ldots,d$,

$$A_4^{1,x}(1_q,i_g,j_g,2_{\bar{q}})\left|\mathcal{M}^0_{q\bar{q}}\right|^2 = \mathrm{Re}\left(\mathcal{M}^{1,x}_{A,4}(p_1,p_i,p_j,p_2)\mathcal{M}^{0,\dagger}_{A,4}(p_1,p_i,p_j,p_2)\right), \qquad (4.33)$$

$$\tilde{A}_4^{1,x}(1_q,3_g,4_g,2_{\bar{q}})\left|\mathcal{M}^0_{q\bar{q}}\right|^2 = \mathrm{Re}\left(\tilde{\mathcal{M}}^{1,x}_{A,4}(p_1,p_3,p_4,p_2)\tilde{\mathcal{M}}^{0,\dagger}_{A,4}(p_1,p_3,p_4,p_2)\right), \qquad (4.34)$$

and

$$\tilde{\mathcal{M}}^{1,x}_{A,4}(p_1,p_3,p_4,p_2) = \mathcal{M}^{1,x}_{A,4}(p_1,p_3,p_4,p_2)+\mathcal{M}^{1,x}_{A,4}(p_1,p_4,p_3,p_2). \qquad (4.35)$$

The renormalised singularity structure of the various contributions can be easily written in terms of the tree-level squared matrix elements multiplied by combinations of infrared singularity operators [55], for which we use the notation defined in [32]. Explicitly, we find

$$\mathcal{P}oles(A_4^{1,a}(1_q,i_g,j_g,2_{\bar{q}})) = 2\left(\mathbf{I}^{(1)}_{qg}(\epsilon,s_{1i})+\mathbf{I}^{(1)}_{gg}(\epsilon,s_{ij})+\mathbf{I}^{(1)}_{g\bar{q}}(\epsilon,s_{j2})\right)A_4^0(1_q,i_g,j_g,2_{\bar{q}}), \qquad (4.36)$$

$$\mathcal{P}oles(A_4^{1,b}(1_q,i_g,j_g,2_{\bar{q}})) = 2\mathbf{I}^{(1)}_{q\bar{q}}(\epsilon,s_{12})A_4^0(1_q,i_g,j_g,2_{\bar{q}}), \qquad (4.37)$$

$$\mathcal{P}oles(A_4^{1,c}(1_q,i_g,j_g,2_{\bar{q}})) = 2\left(\mathbf{I}^{(1)}_{qg,F}(\epsilon,s_{1i})+\mathbf{I}^{(1)}_{gg,F}(\epsilon,s_{ij})+\mathbf{I}^{(1)}_{g\bar{q},F}(\epsilon,s_{j2})\right)A_4^0(1_q,i_g,j_g,2_{\bar{q}}), \qquad (4.38)$$

$$\begin{aligned}
\mathcal{P}oles(\tilde{A}_4^{1,a}(1_q,3_g,4_g,2_{\bar{q}})) &= \Big(2\mathbf{I}^{(1)}_{gg}(\epsilon,s_{34})+\mathbf{I}^{(1)}_{qg}(\epsilon,s_{14})+\mathbf{I}^{(1)}_{g\bar{q}}(\epsilon,s_{23}) \qquad (4.39)\\
&\quad +\mathbf{I}^{(1)}_{qg}(\epsilon,s_{13})+\mathbf{I}^{(1)}_{g\bar{q}}(\epsilon,s_{24})\Big)\tilde{A}_4^0(1_q,3_g,4_g,2_{\bar{q}}),
\end{aligned}$$

$$\mathcal{P}oles(\tilde{A}_4^{1,b}(1_q,3_g,4_g,2_{\bar{q}})) = 2\mathbf{I}^{(1)}_{q\bar{q}}(\epsilon,s_{12})\tilde{A}_4^0(1_q,3_g,4_g,2_{\bar{q}}), \qquad (4.40)$$

$$\begin{aligned}
\mathcal{P}oles(\tilde{A}_4^{1,c}(1_q,3_g,4_g,2_{\bar{q}})) &= \Bigg(2\mathbf{I}^{(1)}_{gg,F}(\epsilon,s_{34})+\mathbf{I}^{(1)}_{qg,F}(\epsilon,s_{14})+\mathbf{I}^{(1)}_{g\bar{q},F}(\epsilon,s_{23})\\
&\quad +\mathbf{I}^{(1)}_{qg,F}(\epsilon,s_{13})+\mathbf{I}^{(1)}_{g\bar{q},F}(\epsilon,s_{24})\Bigg)\times\tilde{A}_4^0(1_q,3_g,4_g,2_{\bar{q}}), \qquad (4.41)
\end{aligned}$$

$$\begin{aligned}
\mathcal{P}oles(\tilde{A}_4^{1,d}(1_q,3_g,4_g,2_{\bar{q}})) &= \Bigg(2\mathbf{I}^{(1)}_{q\bar{q}}(\epsilon,s_{12})+2\mathbf{I}^{(1)}_{gg}(\epsilon,s_{34})-\mathbf{I}^{(1)}_{qg}(\epsilon,s_{14})-\mathbf{I}^{(1)}_{g\bar{q}}(\epsilon,s_{23})\\
&\quad -\mathbf{I}^{(1)}_{qg}(\epsilon,s_{13})-\mathbf{I}^{(1)}_{g\bar{q}}(\epsilon,s_{24})\Bigg)\times\tilde{A}_4^0(1_q,3_g,4_g,2_{\bar{q}}). \qquad (4.42)
\end{aligned}$$

As at tree-level, the one-loop amplitude for

$$\gamma^*(q) \to q(p_1)\bar{q}(p_2)q'(p_3)\bar{q}'(p_4)$$

contains two colour structures,

$$\begin{aligned} M^1_{q\bar{q}q'\bar{q}'} = & \, ie_1 g^2 \left(\frac{g^2}{16\pi^2}\right) \delta_{q_1q_2}\delta_{q_3q_4}\Bigg[\delta_{i_1i_4}\delta_{i_3i_2}\Bigg(N\mathcal{M}^{1,a}_{B,4}(p_1,p_2,p_3,p_4) \\ & -\frac{1}{N}\mathcal{M}^{1,b}_{B,4}(p_1,p_2,p_3,p_4) + N_F\mathcal{M}^{1,c}_{B,4}(p_1,p_2,p_3,p_4)\Bigg) \\ & -\frac{1}{N}\delta_{i_1i_2}\delta_{i_3i_4}\Bigg(N\mathcal{M}^{1,d}_{B,4}(p_1,p_2,p_3,p_4) - \frac{1}{N}\mathcal{M}^{1,e}_{B,4}(p_1,p_2,p_3,p_4) \\ & +N_F\mathcal{M}^{1,f}_{B,4}(p_1,p_2,p_3,p_4)\Bigg)\Bigg] + (1 \leftrightarrow 3, 2 \leftrightarrow 4)\ , \end{aligned} \tag{4.43}$$

where

$$\mathcal{M}^{1,a}_{B,4}(p_1,p_2,p_3,p_4) + \mathcal{M}^{1,e}_{B,4}(p_1,p_2,p_3,p_4) = \mathcal{M}^{1,b}_{B,4}(p_1,p_2,p_3,p_4) + \mathcal{M}^{1,d}_{B,4}(p_1,p_2,p_3,p_4). \tag{4.44}$$

As before, the identical quark matrix element is obtained by permuting the antiquark momenta,

$$M^1_{q\bar{q}q\bar{q}} = M^1_{q\bar{q}q'\bar{q}'} - (2 \leftrightarrow 4). \tag{4.45}$$

Summing over flavours and including symmetry factors, we find that the "squared" matrix element, is given by

$$\begin{aligned} 2\left|M^{0,\dagger}_{4q}M^1_{4q}\right| = & \sum_{q,q'} 2\left|M^{0,\dagger}_{q\bar{q}q'\bar{q}'}M^1_{q\bar{q}q'\bar{q}'}\right| + \sum_q 2\left|M^{0,\dagger}_{q\bar{q}q\bar{q}}M^1_{q\bar{q}q\bar{q}}\right| \\ = & \, N_4\left(\frac{\alpha_s}{2\pi}\right)\Bigg[NN_F B^{1,a}_4(1_q,3_q,4_{\bar{q}},2_{\bar{q}}) - \frac{N_F}{N}B^{1,b}_4(1_q,3_q,4_{\bar{q}},2_{\bar{q}}) + N_F^2 B^{1,c}_4(1_q,3_q,4_{\bar{q}},2_{\bar{q}}) \\ & -C^{1,d}_4(1_q,3_q,4_{\bar{q}},2_{\bar{q}}) + \frac{1}{N^2}C^{1,e}_4(1_q,3_q,4_{\bar{q}},2_{\bar{q}}) - \frac{N_F}{N}C^{1,f}_4(1_q,3_q,4_{\bar{q}},2_{\bar{q}}) \\ & -C^{1,d}_4(2_{\bar{q}},4_{\bar{q}},3_q,1_q) + \frac{1}{N^2}C^{1,e}_4(2_{\bar{q}},4_{\bar{q}},3_q,1_q) - \frac{N_F}{N}C^{1,f}_4(2_{\bar{q}},4_{\bar{q}},3_q,1_q) \\ & +NN_{F,\gamma}\hat{B}^{1,a}_4(1_q,3_q,4_{\bar{q}},2_{\bar{q}}) - \frac{N_{F,\gamma}}{N}\hat{B}^{1,b}_4(1_q,3_q,4_{\bar{q}},2_{\bar{q}}) + N_F N_{F,\gamma}\hat{B}^{1,c}_4(1_q,3_q,4_{\bar{q}},2_{\bar{q}})\Bigg], \end{aligned} \tag{4.46}$$

where for $x = a, b, c$

$$B^{1,x}_4(1_q,3_q,4_{\bar{q}},2_{\bar{q}})\left|\mathcal{M}^0_{q\bar{q}}\right|^2 = \mathrm{Re}\left(\mathcal{M}^{1,x}_{B,4}(p_1,p_2,p_3,p_4)\mathcal{M}^{0,\dagger}_{B,4}(p_1,p_2,p_3,p_4)\right), \tag{4.47}$$

$$\hat{B}^{1,x}_4(1_q,3_q,4_{\bar{q}},2_{\bar{q}})\left|\mathcal{M}^0_{q\bar{q}}\right|^2 = \mathrm{Re}\left(\mathcal{M}^{1,x}_{B,4}(p_1,p_2,p_3,p_4)\mathcal{M}^{0,\dagger}_{B,4}(p_3,p_4,p_1,p_2)\right), \tag{4.48}$$

and for $x = d, e, f$

$$C^{1,x}_4(1_q,3_q,4_{\bar{q}},2_{\bar{q}})\left|\mathcal{M}^0_{q\bar{q}}\right|^2 = -\mathrm{Re}\left(\mathcal{M}^{1,x}_{B,4}(p_1,p_2,p_3,p_4)\mathcal{M}^{0,\dagger}_{B,4}(p_1,p_4,p_3,p_2)\right). \tag{4.49}$$

Using the infrared singularity operators of [55], we can extract the singular contributions of the renormalised one-loop contribution as,

$$\mathcal{P}oles(B_4^{1,a}(1_q,3_q,4_{\bar{q}},2_{\bar{q}})) = 2\left(\mathbf{I}^{(1)}_{q\bar{q}}(\epsilon,s_{14})+\mathbf{I}^{(1)}_{q\bar{q}}(\epsilon,s_{23})\right) B_4^0(1_q,3_q,4_{\bar{q}},2_{\bar{q}}), \tag{4.50}$$

$$\begin{aligned}\mathcal{P}oles(B_4^{1,b}(1_q,3_q,4_{\bar{q}},2_{\bar{q}})) = 2\Big(&2\mathbf{I}^{(1)}_{q\bar{q}}(\epsilon,s_{14})-2\mathbf{I}^{(1)}_{q\bar{q}}(\epsilon,s_{13})+2\mathbf{I}^{(1)}_{q\bar{q}}(\epsilon,s_{23})-2\mathbf{I}^{(1)}_{q\bar{q}}(\epsilon,s_{24})\\ &+\mathbf{I}^{(1)}_{q\bar{q}}(\epsilon,s_{12})+\mathbf{I}^{(1)}_{q\bar{q}}(\epsilon,s_{34})\Big)\times B_4^0(1_q,3_q,4_{\bar{q}},2_{\bar{q}}),\end{aligned} \tag{4.51}$$

$$\mathcal{P}oles(C_4^{1,d}(1_q,3_q,4_{\bar{q}},2_{\bar{q}})) = 2\left(\mathbf{I}^{(1)}_{q\bar{q}}(\epsilon,s_{13})+\mathbf{I}^{(1)}_{q\bar{q}}(\epsilon,s_{24})\right) C_4^0(1_q,3_q,4_{\bar{q}},2_{\bar{q}}), \tag{4.52}$$

$$\begin{aligned}\mathcal{P}oles(C_4^{1,e}(1_q,3_q,4_{\bar{q}},2_{\bar{q}})) = 2\Big(&\mathbf{I}^{(1)}_{q\bar{q}}(\epsilon,s_{12})+\mathbf{I}^{(1)}_{q\bar{q}}(\epsilon,s_{14})+\mathbf{I}^{(1)}_{q\bar{q}}(\epsilon,s_{23})+\mathbf{I}^{(1)}_{q\bar{q}}(\epsilon,s_{34})\\ &-\mathbf{I}^{(1)}_{q\bar{q}}(\epsilon,s_{13})-\mathbf{I}^{(1)}_{q\bar{q}}(\epsilon,s_{24})\Big)\times C_4^0(1_q,3_q,4_{\bar{q}},2_{\bar{q}}),\end{aligned} \tag{4.53}$$

$$\mathcal{P}oles(\hat{B}_4^{1,a}(1_q,3_q,4_{\bar{q}},2_{\bar{q}})) = 2\left(\mathbf{I}^{(1)}_{q\bar{q}}(\epsilon,s_{14})+\mathbf{I}^{(1)}_{q\bar{q}}(\epsilon,s_{23})\right) \hat{B}_4^0(1_q,3_q,4_{\bar{q}},2_{\bar{q}}), \tag{4.54}$$

$$\begin{aligned}\mathcal{P}oles(\hat{B}_4^{1,b}(1_q,3_q,4_{\bar{q}},2_{\bar{q}})) = 2\Big(&2\mathbf{I}^{(1)}_{q\bar{q}}(\epsilon,s_{14})-2\mathbf{I}^{(1)}_{q\bar{q}}(\epsilon,s_{13})+2\mathbf{I}^{(1)}_{q\bar{q}}(\epsilon,s_{23})-2\mathbf{I}^{(1)}_{q\bar{q}}(\epsilon,s_{24})\\ &+\mathbf{I}^{(1)}_{q\bar{q}}(\epsilon,s_{12})+\mathbf{I}^{(1)}_{q\bar{q}}(\epsilon,s_{34})\Big)\times \hat{B}_4^0(1_q,3_q,4_{\bar{q}},2_{\bar{q}})\,.\end{aligned} \tag{4.55}$$

4.3 Two-loop matrix elements for three partons

The renormalised two-loop amplitude $M^2_{q\bar{q}g}$ for a virtual photon to produce a quark-antiquark pair together with a single gluon,

$$\gamma^*(q)\to q(p_1)\bar{q}(p_2)g(p_3)$$

contains a single colour structure such that

$$M^2_{q\bar{q}g} = ie\sqrt{2}g\left(\frac{g^2}{16\pi^2}\right)^2 T^{a_3}_{i_1 i_2}\mathcal{M}^2_{A,3}(p_1,p_3,p_2)\,. \tag{4.56}$$

At NNLO, there are two contributions. One from the interference of the two-loop and tree-level amplitudes (4.3), the other from the square of the one-loop amplitudes given in (4.26). These terms were computed in [18] by reducing the large number of two-loop Feynman integrals to a small number of master integrals, using integration-by-parts [56] and Lorentz-invariance [57] identities, solved using the Laporta algorithm [58]. The relevant master integrals (two-loop four-point functions with one off-shell leg) were then derived [59] from their differential equations [57, 60, 61].

The resulting virtual three-parton contributions are given by,

$$2\mathrm{Re}\left(M^{0,\dagger}_{q\bar{q}g}M^2_{q\bar{q}g}\right) = N_3\left(\frac{\alpha_s}{2\pi}\right)^2 A_3^{(2\times 0)}(1_q,3_g,2_{\bar{q}})\,, \tag{4.57}$$

$$\mathrm{Re}\left(M^{1,\dagger}_{q\bar{q}g}M^1_{q\bar{q}g}\right) = N_3\left(\frac{\alpha_s}{2\pi}\right)^2 A_3^{(1\times 1)}(1_q,3_g,2_{\bar{q}})\,. \tag{4.58}$$

Following [55], we organise the infrared pole structure of the NNLO contributions renormalised in the $\overline{\rm MS}$ scheme in terms of the tree and renormalised one-loop amplitudes such that,

$$
\begin{aligned}
\mathcal{P}oles\left(A_3^{(2\times 0)}(1_q,3_g,2_{\bar q}) + A_3^{(1\times 1)}(1_q,3_g,2_{\bar q})\right) &= 2\Bigg[-\left(\boldsymbol{I}^{(1)}_{q\bar q g}(\epsilon)\right)^2 - \frac{\beta_0}{\epsilon}\boldsymbol{I}^{(1)}_{q\bar q g}(\epsilon) \\
&\quad + e^{-\epsilon\gamma}\frac{\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)}\left(\frac{\beta_0}{\epsilon}+K\right)\boldsymbol{I}^{(1)}_{q\bar q g}(2\epsilon) + \boldsymbol{H}^{(2)}_{q\bar q g}\Bigg] A_3^0(1_q,3_g,2_{\bar q}) \\
&\quad + 2\,\boldsymbol{I}^{(1)}_{q\bar q g}(\epsilon)\, A_3^{(1\times 0)}(1_q,3_g,2_{\bar q})\,.
\end{aligned}
\tag{4.59}
$$

Here,

$$
\begin{aligned}
\boldsymbol{I}^{(1)}_{q\bar q g}(\epsilon) &= N\left(\boldsymbol{I}^{(1)}_{qg}(\epsilon,s_{13}) + \boldsymbol{I}^{(1)}_{qg}(\epsilon,s_{23})\right) - \frac{1}{N}\boldsymbol{I}^{(1)}_{q\bar q}(\epsilon,s_{12}) \\
&\quad + N_F\left(\boldsymbol{I}^{(1)}_{qg,F}(\epsilon,s_{13}) + \boldsymbol{I}^{(1)}_{qg,F}(\epsilon,s_{23})\right),
\end{aligned}
\tag{4.60}
$$

with the individual $\boldsymbol{I}^{(1)}_{ij}$ defined in [32] and

$$
\begin{aligned}
\boldsymbol{H}^{(2)}_{q\bar q g} &= \frac{e^{\epsilon\gamma}}{4\,\epsilon\,\Gamma(1-\epsilon)}\Bigg[\left(4\zeta_3+\frac{589}{432}-\frac{11\pi^2}{72}\right)N^2 + \left(-\frac{1}{2}\zeta_3-\frac{41}{54}-\frac{\pi^2}{48}\right) \\
&\quad + \left(-3\zeta_3-\frac{3}{16}+\frac{\pi^2}{4}\right)\frac{1}{N^2} + \left(-\frac{19}{18}+\frac{\pi^2}{36}\right)NN_F + \left(-\frac{1}{54}-\frac{\pi^2}{24}\right)\frac{N_F}{N} + \frac{5}{27}N_F^2.\Bigg]\,.
\end{aligned}
\tag{4.61}
$$

We denote the finite contributions as,

$$
\begin{aligned}
\mathcal{F}inite(A_3^{(2\times 0)}(1_q,3_g,2_{\bar q})) &= N^2 A^{(2\times 0),finite}_{3,N^2} + A^{(2\times 0),finite}_{3,1} + \frac{1}{N^2}A^{(2\times 0),finite}_{3,1/N^2} \\
&\quad + NN_F A^{(2\times 0),finite}_{3,NN_F} + \frac{N_F}{N}A^{(2\times 0),finite}_{3,N_F/N} \\
&\quad + N_F^2 A^{(2\times 0),finite}_{3,N_F^2} + N_{F,\gamma}\left(\frac{4}{N}-N\right)A^{(2\times 0),finite}_{3,N_{F,\gamma}},
\end{aligned}
\tag{4.62}
$$

$$
\begin{aligned}
\mathcal{F}inite(A_3^{(1\times 1)}(1_q,3_g,2_{\bar q})) &= N^2 A^{(1\times 1),finite}_{3,N^2} + A^{(1\times 1),finite}_{3,1} + \frac{1}{N^2}A^{(1\times 1),finite}_{3,1/N^2} \\
&\quad + NN_F A^{(1\times 1),finite}_{3,NN_F} + \frac{N_F}{N}A^{(1\times 1),finite}_{3,N_F/N} \\
&\quad + N_F^2 A^{(1\times 1),finite}_{3,N_F^2}.
\end{aligned}
\tag{4.63}
$$

Explicit formulae for the finite remainders have been given in [18]. These are expressed in terms of one-dimensional and two-dimensional harmonic polylogarithms (HPLs and 2dHPLs) [62, 59], which are generalisations of the well-known Nielsen polylogarithms [63]. A numerical implementation, which is required for all practical applications, is available for HPLs and 2dHPLs [64].

Finally, a finite NNLO contribution arises from the squared one-loop amplitude (4.29) for $\gamma^* \to ggg$:

$$
C_3^{(1\times 1)}(1_g,2_g,3_g) = N_{F,\gamma}\left(\frac{4}{N}-N\right)C^{(1\times 1),finite}_{3,N_{F,\gamma}}\,.
\tag{4.64}
$$

5. Construction of the NLO subtraction term

Three-jet production at the leading order is given by:

$$\mathrm{d}\sigma_{\mathrm{LO}}^{R} = N_3 \, \mathrm{d}\Phi_3(p_1,p_2,p_3;q)\, A_3^0(1_q,3_g,2_{\bar{q}})\, J_3^{(3)}(p_1,p_2,p_3)\,. \tag{5.1}$$

This leading order cross section defines the normalisation for all higher order corrections discussed in the following.

At NLO, the tree-level four-parton processes $\gamma^* \to q\bar{q}gg$, $\gamma^* \to q\bar{q}q'\bar{q}'$ (non-identical quarks) and $\gamma^* \to q\bar{q}q\bar{q}$ (identical quarks) yield three-jet final states. Only the two former processes require subtraction, since the third process is infrared finite.

The four-parton real radiation contribution to the NLO cross section is

$$\begin{aligned}\mathrm{d}\sigma_{\mathrm{NLO}}^{R} &= N_4\, \mathrm{d}\Phi_4(p_1,\ldots,p_4;q)\\ &\times\Bigg\{\frac{N}{2}\sum_{(i,j)\in P(3,4)} A_4^0(1_q,i_g,j_g,2_{\bar{q}}) - \frac{1}{2N}\tilde{A}_4^0(1_q,3_g,4_g,2_{\bar{q}}) + N_F B_4^0(1_q,3_q,4_{\bar{q}},2_{\bar{q}})\\ &-\frac{1}{N}\left(C_4^0(1_q,3_q,4_{\bar{q}},2_{\bar{q}}) + C_4^0(2_{\bar{q}},4_{\bar{q}},3_q,1_q)\right)\Bigg\} J_3^{(4)}(p_1,\ldots,p_4)\,.\end{aligned} \tag{5.2}$$

The antenna subtraction term is then constructed as:

$$\begin{aligned}\mathrm{d}\sigma_{\mathrm{NLO}}^{S} &= N_4\, \mathrm{d}\Phi_4(p_1,\ldots,p_4;q)\\ &\times\Bigg\{\sum_{(i,j)\in P(3,4)}\Bigg[\frac{N}{2}\, d_3^0(1_q,i_g,j_g)\, A_3^0(\widetilde{(1i)}_q,\widetilde{(ji)}_g,2_{\bar{q}})\, J_3^{(3)}(\widetilde{p_{1i}},\widetilde{p_{ji}},p_2)\\ &+\frac{N}{2}\, d_3^0(2_{\bar{q}},i_g,j_g)\, A_3^0(1_q,\widetilde{(ji)}_g,\widetilde{(2i)}_{\bar{q}})\, J_3^{(3)}(p_1,\widetilde{p_{ji}},\widetilde{p_{2i}})\\ &-\frac{1}{2N}\, A_3^0(1_q,i_g,2_{\bar{q}})\, A_3^0(\widetilde{(1i)}_q,j_g,\widetilde{(2i)}_{\bar{q}})\, J_3^{(3)}(\widetilde{p_{1i}},p_j,\widetilde{p_{2i}})\Bigg]\\ &+N_F\,\frac{1}{2}\Bigg[E_3^0(1_q,3_{q'},4_{\bar{q}'})\, A_3^0(\widetilde{(13)}_q,\widetilde{(43)}_g,2_{\bar{q}})\, J_3^{(3)}(\widetilde{p_{13}},\widetilde{p_{43}},p_2)\\ &+E_3^0(2_{\bar{q}},3_{q'},4_{\bar{q}'})\, A_3^0(1_q,\widetilde{(34)}_g,\widetilde{(24)}_{\bar{q}})\, J_3^{(3)}(p_1,\widetilde{p_{34}},\widetilde{p_{24}})\Bigg]\Bigg\}\,.\end{aligned} \tag{5.3}$$

Integration of this subtraction term over the antenna phase spaces yields:

$$\begin{aligned}\mathrm{d}\sigma_{\mathrm{NLO}}^{S} &= N_3\left(\frac{\alpha_s}{2\pi}\right)\mathrm{d}\Phi_3(p_1,\ldots,p_3;q)\\ &\times\left\{\frac{N}{2}\,\left[\mathcal{D}_3^0(s_{13})+\mathcal{D}_3^0(s_{23})\right] - \frac{1}{N}\mathcal{A}_3^0(s_{12}) + \frac{N_F}{2}\,\left[\mathcal{E}_3^0(s_{13})+\mathcal{E}_3^0(s_{23})\right]\right\}\\ &\times A_3^0(1_q,3_g,2_{\bar{q}})\, J_3^{(3)}(p_1,p_2,p_3)\end{aligned} \tag{5.4}$$

Together with the virtual one-loop contribution to $\gamma^* \to q\bar{q}g$,

$$
\begin{aligned}
\mathrm{d}\sigma^V_{\mathrm{NLO}} &= N_3 \left(\frac{\alpha_s}{2\pi}\right) \mathrm{d}\Phi_3(p_1,\ldots,p_3;q)\, J_3^{(3)}(p_1,p_2,p_3) \\
&\quad \times \Bigg(N\left[A_3^1(1_q,3_g,2_{\bar{q}}) + \mathcal{A}_2^1(s_{123})A_3^0(1_q,3_g,2_{\bar{q}})\right] \\
&\quad - \frac{1}{N}\left[\tilde{A}_3^1(1_q,3_g,2_{\bar{q}}) + \mathcal{A}_2^1(s_{123})A_3^0(1_q,3_g,2_{\bar{q}})\right] + N_F \hat{A}_3^1(1_q,3_g,2_{\bar{q}})\Bigg)\,, \qquad (5.5)
\end{aligned}
$$

one obtains

$$
\mathcal{P}oles\left(\mathrm{d}\sigma^S_{\mathrm{NLO}}\right) + \mathcal{P}oles\left(\mathrm{d}\sigma^V_{\mathrm{NLO}}\right) = 0\,, \qquad (5.6)
$$

thus yielding an infrared-finite result.

6. Construction of the N^2 colour factor

The N^2 colour factor receives contributions from five-parton tree-level $\gamma^* \to q\bar{q}ggg$, four-parton one-loop $\gamma^* \to q\bar{q}gg$ and tree-level two-loop $\gamma^* \to q\bar{q}g$. The multiple gluon emissions are colour-ordered, and the squared matrix elements do not contain interference amplitudes between different orderings. In the loop contributions to this colour factor, non-planar momentum arrangements are absent.

6.1 Five-parton contribution

At leading colour, the five parton contribution to three-jet final states arises from the colour-ordered emission of three gluons in $\gamma^* \to q\bar{q}ggg$. The matrix element for one ordering is:

$$
\left|M^0_{q\bar{q}3g}\right|^2 = N_5\, N^2 A_5^0(1_q,i_g,j_g,k_g,2_{\bar{q}})\,. \qquad (6.1)
$$

The real radiation contribution to the cross section is obtained by averaging over all possible six orderings:

$$
\begin{aligned}
\mathrm{d}\sigma^R_{\mathrm{NNLO},N^2} &= N_5\, N^2\, \mathrm{d}\Phi_5(p_1,\ldots,p_5;q)\, \frac{1}{3!} \sum_{(i,j,k)\in P(3,4,5)} A_5^0(1_q,i_g,j_g,k_g,2_{\bar{q}}) J_3^{(5)}(p_1,\ldots,p_5) \\
&= N_5\, N^2\, \mathrm{d}\Phi_5(p_1,\ldots,p_5;q)\, \frac{1}{3!} \\
&\quad \sum_{(i,j,k)\in P_C(3,4,5)} \left[A_5^0(1_q,i_g,j_g,k_g,2_{\bar{q}}) + A_5^0(1_q,k_g,j_g,i_g,2_{\bar{q}})\right] J_3^{(5)}(p_1,\ldots,p_5)\,, \qquad (6.2)
\end{aligned}
$$

where the second expression is obtained by restricting the summation to the three cyclic permutations of the gluon momenta, while making the corresponding three non-cyclic permutations explicit. The cyclic form (6.2) is more appropriate for the construction of the real radiation subtraction term, since this form matches onto the full quark-gluon antenna functions of [32], which have a cyclic ambiguity in their momentum arrangements.

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The real radiation subtraction term for this colour factor reads

$$
\begin{aligned}
\mathrm{d}\sigma^{S}_{\mathrm{NNLO},N^2} &= N_5\, N^2\, \mathrm{d}\Phi_5(p_1,\ldots,p_5;q)\,\frac{1}{3!} \sum_{(i,j,k)\in P_C(3,4,5)} \Bigg\{ \\
& d_3^0(1_q,i_g,j_g)\, A_4^0(\widetilde{(1i)}_q,\widetilde{(ji)}_g,k_g,2_{\bar q})\, J_3^{(4)}(\widetilde{p_{1i}},\widetilde{p_{ji}},p_k,p_2) \\
& +f_3^0(i_g,j_g,k_g)\, A_4^0(1_q,\widetilde{(ij)}_g,\widetilde{(kj)}_g,2_{\bar q})\, J_3^{(4)}(p_1,\widetilde{p_{ij}},\widetilde{p_{kj}},p_2) \\
& +d_3^0(2_{\bar q},k_g,j_g)\, A_4^0(1_q,i_g,\widetilde{(jk)}_g,\widetilde{(2k)}_{\bar q})\, J_3^{(4)}(p_1,p_i,\widetilde{p_{jk}},\widetilde{p_{2k}}) \\
& +d_3^0(1_q,k_g,j_g)\, A_4^0(\widetilde{(1k)}_q,\widetilde{(jk)}_g,i_g,2_{\bar q})\, J_3^{(4)}(\widetilde{p_{1k}},\widetilde{p_{jk}},p_i,p_2) \\
& +f_3^0(k_g,j_g,i_g)\, A_4^0(1_q,\widetilde{(kj)}_g,\widetilde{(ij)}_g,2_{\bar q})\, J_3^{(4)}(p_1,\widetilde{p_{kj}},\widetilde{p_{ij}},p_2) \\
& +d_3^0(2_{\bar q},i_g,j_g)\, A_4^0(1_q,k_g,\widetilde{(ji)}_g,\widetilde{(2i)}_{\bar q})\, J_3^{(4)}(p_1,p_k,\widetilde{p_{ji}},\widetilde{p_{2i}}) \\
& +\Bigg(D_{4,a}^0(1_q,i_g,j_g,k_g) - d_3^0(1_q,i_g,j_g)\, d_3^0(\widetilde{(1i)}_q,\widetilde{(ji)}_g,k_g) \\
& -f_3^0(i_g,j_g,k_g)\, d_3^0(1_q,\widetilde{(ij)}_g,\widetilde{(kj)}_g)\Bigg) A_3^0(\widetilde{(1ij)}_q,\widetilde{(kji)}_g,2_{\bar q})\, J_3^{(3)}(\widetilde{p_{1ij}},\widetilde{p_{kji}},p_2) \\
& +\Bigg(D_{4,b}^0(1_q,i_g,j_g,k_g) - f_3^0(i_g,j_g,k_g)\, d_3^0(1_q,\widetilde{(kj)}_g,\widetilde{(ij)}_g) \\
& -d_3^0(1_q,k_g,j_g)\, d_3^0(\widetilde{(1k)}_q,\widetilde{(jk)}_g,i_g)\Bigg) A_3^0(\widetilde{(1kj)}_q,\widetilde{(ijk)}_g,2_{\bar q})\, J_3^{(3)}(\widetilde{p_{1kj}},\widetilde{p_{ijk}},p_2) \\
& +\Bigg(D_{4,c}^0(1_q,i_g,j_g,k_g) \\
& -d_3^0(1_q,i_g,j_g)\, d_3^0(\widetilde{(1i)}_q,k_g,\widetilde{(ji)}_g)\Bigg) A_3^0(\widetilde{(1ik)}_q,\widetilde{(jki)}_g,2_{\bar q})\, J_3^{(3)}(\widetilde{p_{1ik}},\widetilde{p_{jki}},p_2) \\
& +\Bigg(D_{4,d}^0(1_q,i_g,j_g,k_g) \\
& -d_3^0(1_q,k_g,j_g)\, d_3^0(\widetilde{(1k)}_q,i_g,\widetilde{(jk)}_g)\Bigg) A_3^0(\widetilde{(1ki)}_q,\widetilde{(jik)}_g,2_{\bar q})\, J_3^{(3)}(\widetilde{p_{1ki}},\widetilde{p_{jik}},p_2) \\
& +\Bigg(D_{4,a}^0(2_q,i_g,j_g,k_g) - d_3^0(2_q,i_g,j_g)\, d_3^0(\widetilde{(2i)}_q,\widetilde{(ji)}_g,k_g) \\
& -f_3^0(i_g,j_g,k_g)\, d_3^0(2_q,\widetilde{(ij)}_g,\widetilde{(kj)}_g)\Bigg) A_3^0(\widetilde{(2ij)}_q,\widetilde{(kji)}_g,1_{\bar q})\, J_3^{(3)}(\widetilde{p_{2ij}},\widetilde{p_{kji}},p_1) \\
& +\Bigg(D_{4,b}^0(2_q,i_g,j_g,k_g) - f_3^0(i_g,j_g,k_g)\, d_3^0(2_q,\widetilde{(kj)}_g,\widetilde{(ij)}_g) \\
& -d_3^0(2_q,k_g,j_g)\, d_3^0(\widetilde{(2k)}_q,\widetilde{(jk)}_g,i_g)\Bigg) A_3^0(\widetilde{(2kj)}_q,\widetilde{(ijk)}_g,1_{\bar q})\, J_3^{(3)}(\widetilde{p_{2kj}},\widetilde{p_{ijk}},p_1)
\end{aligned}
$$

$$
\begin{aligned}
&+\Bigg(D_{4,c}^0(2_q,i_g,j_g,k_g)\\
&-d_3^0(2_q,i_g,j_g)\,d_3^0(\widetilde{(2i)}_q,k_g,\widetilde{(ji)}_g)\Bigg)A_3^0(\widetilde{(2ik)}_q,\widetilde{(jki)}_g,1_{\bar q})\,J_3^{(3)}(\widetilde{p_{2ik}},\widetilde{p_{jki}},p_1)\\
&+\Bigg(D_{4,d}^0(2_q,i_g,j_g,k_g)\\
&-d_3^0(2_q,k_g,j_g)\,d_3^0(\widetilde{(2k)}_q,i_g,\widetilde{(jk)}_g)\Bigg)A_3^0(\widetilde{(2ki)}_q,\widetilde{(jik)}_g,1_{\bar q})\,J_3^{(3)}(\widetilde{p_{2ki}},\widetilde{p_{jik}},p_1)\\
&-\Bigg(\tilde A_4^0(1_q,i_g,k_g,2_{\bar q})-A_3^0(1_q,i_g,2_{\bar q})A_3^0(\widetilde{(1i)}_q,k_g,\widetilde{(2i)}_{\bar q})\\
&-A_3^0(1_q,k_g,2_{\bar q})A_3^0(\widetilde{(1k)}_q,i_g,\widetilde{(2k)}_{\bar q})\Bigg)\,A_3^0(\widetilde{(1ik)}_q,j_g,\widetilde{(2ki)}_{\bar q})\,J_3^{(3)}(\widetilde{p_{1ik}},p_j,\widetilde{p_{2ki}})\\
&-\frac{1}{2}d_3^0(1_q,i_g,j_g)d_3^0(2_{\bar q},k_g,\widetilde{(ji)}_g)\,A_3^0(\widetilde{(1i)}_q,\widetilde{((ji)k)}_g,\widetilde{(2k)}_{\bar q})\,J_3^{(3)}(\widetilde{p_{1i}},\widetilde{p_{(ji)k}},\widetilde{p_{2k}})\\
&-\frac{1}{2}d_3^0(2_{\bar q},k_g,j_g)d_3^0(1_q,i_g,\widetilde{(jk)}_g)\,A_3^0(\widetilde{(1i)}_q,\widetilde{((jk)i)}_g,\widetilde{(2k)}_{\bar q})\,J_3^{(3)}(\widetilde{p_{1i}},\widetilde{p_{(jk)i}},\widetilde{p_{2k}})\\
&-\frac{1}{2}d_3^0(1_q,k_g,j_g)d_3^0(2_{\bar q},i_g,\widetilde{(jk)}_g)\,A_3^0(\widetilde{(1k)}_q,\widetilde{((jk)i)}_g,\widetilde{(2i)}_{\bar q})\,J_3^{(3)}(\widetilde{p_{1k}},\widetilde{p_{(jk)i}},\widetilde{p_{2i}})\\
&-\frac{1}{2}d_3^0(2_{\bar q},i_g,j_g)d_3^0(1_q,k_g,\widetilde{(ji)}_g)\,A_3^0(\widetilde{(1k)}_q,\widetilde{((ji)k)}_g,\widetilde{(2i)}_{\bar q})\,J_3^{(3)}(\widetilde{p_{1k}},\widetilde{p_{(ji)k}},\widetilde{p_{2i}})\\
&+\frac{1}{2}d_3^0(1_q,i_g,j_g)d_3^0(\widetilde{(1i)}_q,k_g,\widetilde{(ji)}_g)\,A_3^0(\widetilde{((1i)k)}_q,\widetilde{((ji)k)}_g,2_{\bar q})\,J_3^{(3)}(\widetilde{p_{(1i)k}},\widetilde{p_{(ji)k}},p_2)\\
&+\frac{1}{2}d_3^0(1_q,k_g,j_g)d_3^0(\widetilde{(1k)}_q,i_g,\widetilde{(jk)}_g)\,A_3^0(\widetilde{((1k)i)}_q,\widetilde{((jk)i)}_g,2_{\bar q})\,J_3^{(3)}(\widetilde{p_{(1k)i}},\widetilde{p_{(jk)i}},p_2)\\
&+\frac{1}{2}d_3^0(2_{\bar q},i_g,j_g)d_3^0(\widetilde{(2i)}_{\bar q},k_g,\widetilde{(ji)}_g)\,A_3^0(1_q,\widetilde{((ji)k)}_g,\widetilde{((2i)k)}_{\bar q})\,J_3^{(3)}(p_1,\widetilde{p_{(ji)k}},\widetilde{p_{(2i)k}})\\
&+\frac{1}{2}d_3^0(2_{\bar q},k_g,j_g)d_3^0(\widetilde{(2k)}_{\bar q},i_g,\widetilde{(jk)}_g)\,A_3^0(1_q,\widetilde{((jk)i)}_g,\widetilde{((2k)i)}_{\bar q})\,J_3^{(3)}(p_1,\widetilde{p_{(jk)i}},\widetilde{p_{(2k)i}})\\
&-\frac{1}{2}A_3^0(1_q,i_g,2_{\bar q})d_3^0(\widetilde{(1i)}_q,k_g,j_g)\,A_3^0(\widetilde{((1i)k)}_q,\widetilde{(jk)}_g,\widetilde{(2i)}_{\bar q})\,J_3^{(3)}(\widetilde{p_{(1i)k}},\widetilde{p_{jk}},\widetilde{p_{2i}})\\
&+\frac{1}{2}d_3^0(1_q,k_g,j_g)A_3^0(\widetilde{(1k)}_q,i_g,2_{\bar q})\,A_3^0(\widetilde{((1k)i)}_q,\widetilde{(jk)}_g,\widetilde{(2i)}_{\bar q})\,J_3^{(3)}(\widetilde{p_{(1k)i}},\widetilde{p_{jk}},\widetilde{p_{2i}})\\
&-\frac{1}{2}A_3^0(1_q,k_g,2_{\bar q})d_3^0(\widetilde{(1k)}_q,i_g,j_g)\,A_3^0(\widetilde{((1k)i)}_q,\widetilde{(ji)}_g,\widetilde{(2k)}_{\bar q})\,J_3^{(3)}(\widetilde{p_{(1k)i}},\widetilde{p_{ji}},\widetilde{p_{2k}})\\
&+\frac{1}{2}d_3^0(1_q,i_g,j_g)A_3^0(\widetilde{(1i)}_q,k_g,2_{\bar q})\,A_3^0(\widetilde{((1i)k)}_q,\widetilde{(ji)}_g,\widetilde{(2k)}_{\bar q})\,J_3^{(3)}(\widetilde{p_{(1i)k}},\widetilde{p_{ji}},\widetilde{p_{2k}})\\
&-\frac{1}{2}A_3^0(1_q,i_g,2_{\bar q})d_3^0(\widetilde{(2i)}_{\bar q},k_g,j_g)\,A_3^0(\widetilde{(1i)}_q,\widetilde{(jk)}_g,\widetilde{((2i)k)}_{\bar q})\,J_3^{(3)}(\widetilde{p_{1i}},\widetilde{p_{jk}},\widetilde{p_{(2i)k}})\\
&+\frac{1}{2}d_3^0(2_{\bar q},k_g,j_g)A_3^0(1_q,i_g,\widetilde{(2k)}_{\bar q})\,A_3^0(\widetilde{(1i)}_q,\widetilde{(jk)}_g,\widetilde{((2k)i)}_{\bar q})\,J_3^{(3)}(\widetilde{p_{1i}},\widetilde{p_{jk}},\widetilde{p_{(2k)i}})\\
&-\frac{1}{2}A_3^0(1_q,k_g,2_{\bar q})d_3^0(\widetilde{(2k)}_{\bar q},i_g,j_g)\,A_3^0(\widetilde{(1k)}_q,\widetilde{(ji)}_g,\widetilde{((2k)i)}_{\bar q})\,J_3^{(3)}(\widetilde{p_{1k}},\widetilde{p_{ji}},\widetilde{p_{(2k)i}})\\
&+\frac{1}{2}d_3^0(2_{\bar q},i_g,j_g)A_3^0(1_q,k_g,\widetilde{(2i)}_{\bar q})\,A_3^0(\widetilde{(1k)}_q,\widetilde{(ji)}_g,\widetilde{((2i)k)}_{\bar q})\,J_3^{(3)}(\widetilde{p_{1k}},\widetilde{p_{ji}},\widetilde{p_{(2i)k}})
\end{aligned}
$$

$$-\frac{1}{2}A_3^0(1_q,k_g,2_{\bar{q}})A_3^0(\widetilde{(1k)}_q,i_g,\widetilde{(2k)}_{\bar{q}})\,A_3^0(\widetilde{(\widetilde{(1k)}i)}_q,j_g,\widetilde{(\widetilde{(2k)}i)}_{\bar{q}})\,J_3^{(3)}(\widetilde{p_{\widetilde{(1k)}i}},p_j,\widetilde{p_{\widetilde{(2k)}i}})$$
$$\left.-\frac{1}{2}A_3^0(1_q,i_g,2_{\bar{q}})A_3^0(\widetilde{(1i)}_q,k_g,\widetilde{(2i)}_{\bar{q}})\,A_3^0(\widetilde{(\widetilde{(1i)}k)}_q,j_g,\widetilde{(\widetilde{(2i)}k)}_{\bar{q}})\,J_3^{(3)}(\widetilde{p_{\widetilde{(1i)}k}},p_j,\widetilde{p_{\widetilde{(2i)}k}})\right\}.$$

6.2 Four-parton contribution

The leading colour four-parton contribution comes from the one-loop correction to $\gamma^* \to q\bar{q}gg$, where the gluonic emissions are colour-ordered. It reads

$$\begin{aligned}\mathrm{d}\sigma_{\mathrm{NNLO},N^2}^{V,1} &= N_4\,N^2\left(\frac{\alpha_s}{2\pi}\right)\mathrm{d}\Phi_4(p_1,\ldots,p_4;q)\\ &\quad\times\frac{1}{2}\sum_{(i,j)\in(3,4)}A_4^{1,a}(1_q,i_g,j_g,2_{\bar{q}})\,J_3^{(4)}(p_1,p_2,p_3,p_4),\end{aligned}\tag{6.3}$$

The one-loop single unresolved subtraction term for this colour factor is

$$\begin{aligned}&\mathrm{d}\sigma_{\mathrm{NNLO},N^2}^{VS,1} = N_4\,N^2\left(\frac{\alpha_s}{2\pi}\right)\mathrm{d}\Phi_4(p_1,\ldots,p_4;q)\\
&\left\{-\frac{1}{2}\sum_{(i,j)\in(3,4)}\left[\frac{1}{2}\mathcal{D}_3^0(s_{1i})+\frac{1}{3}\mathcal{F}_3^0(s_{ij})+\frac{1}{2}\mathcal{D}_3^0(s_{2j})\right]\,A_4^0(1_q,i_g,j_g,2_{\bar{q}})\,J_3^{(4)}(p_1,\ldots,p_4)\right.\\
&+\left\{\frac{1}{2}\sum_{(i,j)\in(3,4)}\left[d_3^0(1_q,i_g,j_g)\left[A_3^1(\widetilde{(1i)}_q,\widetilde{(ji)}_g,2_{\bar{q}})+\mathcal{A}_2^1(s_{1234})A_3^0(\widetilde{(1i)}_q,\widetilde{(ji)}_g,2_{\bar{q}})\right]\right.\right.\\
&+d_3^1(1_q,i_g,j_g)\,A_3^0(\widetilde{(1i)}_q,\widetilde{(ji)}_g,2_{\bar{q}})\\
&+\frac{1}{2}\left(\mathcal{D}_3^0(s_{1ij})+\mathcal{D}_3^0(s_{2\widetilde{(ji)}})\right)\,d_3^0(1_q,i_g,j_g)\,A_3^0(\widetilde{(1i)}_q,\widetilde{(ji)}_g,2_{\bar{q}})\\
&+\left(\frac{1}{2}\mathcal{D}_3^0(s_{1i})+\frac{1}{3}\mathcal{F}_3^0(s_{ij})+\frac{1}{2}\mathcal{D}_3^0(s_{1j})-\mathcal{D}_3^0(s_{1ij})\right)\,d_3^0(1_q,i_g,j_g)\,A_3^0(\widetilde{(1i)}_q,\widetilde{(ji)}_g,2_{\bar{q}})\\
&+b_0\log\frac{q^2}{s_{1ij}}d_3^0(1_q,i_g,j_g)\,A_3^0(\widetilde{(1i)}_q,\widetilde{(ji)}_g,2_{\bar{q}})\\
&-\frac{1}{2}\left(\frac{1}{2}\mathcal{D}_3^0(s_{2\widetilde{(ji)}})-\frac{1}{2}\mathcal{D}_3^0(s_{1ij})+\mathcal{A}_3^0(s_{\widetilde{(1i)}2})-\frac{1}{2}\mathcal{D}_3^0(s_{2j})+\frac{1}{2}\mathcal{D}_3^0(s_{1j})-\mathcal{A}_3^0(s_{12})\right)\\
&\left.\left.d_3^0(1_q,i_g,j_g)\,A_3^0(\widetilde{(1i)}_q,\widetilde{(ji)}_g,2_{\bar{q}})\right]J_3^{(3)}(\widetilde{p_{1i}},\widetilde{p_{ji}},p_2)+(1\leftrightarrow 2)\right\}\\
&-\frac{1}{2}\sum_{(i,j)\in(3,4)}\left[\tilde{A}_3^1(1_q,i_g,2_{\bar{q}})A_3^0(\widetilde{(1i)}_q,j_g,\widetilde{(2i)}_{\bar{q}})\right.\\
&+\left(\mathcal{A}_3^0(s_{12})-\mathcal{A}_3^0(s_{12i})\right)\,A_3^0(1_q,i_g,2_{\bar{q}})\,A_3^0(\widetilde{(1i)}_q,j_g,\widetilde{(2i)}_{\bar{q}})\\
&+\frac{1}{2}\left(\mathcal{A}_3^0(s_{12i})-\frac{1}{2}\mathcal{D}_3^0(s_{\widetilde{(1i)}j})-\frac{1}{2}\mathcal{D}_3^0(s_{\widetilde{(2i)}j})-\mathcal{A}_3^0(s_{12})+\frac{1}{2}\mathcal{D}_3^0(s_{1j})+\frac{1}{2}\mathcal{D}_3^0(s_{2j})\right)\\
&\left.A_3^0(1_q,i_g,2_{\bar{q}})\,A_3^0(\widetilde{(1i)}_q,j_g,\widetilde{(2i)}_{\bar{q}})\right]J_3^{(3)}(\widetilde{p_{1i}},\widetilde{p_{2i}},p_j)\end{aligned}\tag{6.4}$$

6.3 Three-parton contribution

The three-parton contribution consists of the two-loop three-parton matrix element together with the integrated forms of the five-parton and four-parton subtraction terms,

$$
\begin{aligned}
&{\rm d}\sigma^{S}_{{\rm NNLO},N^2} + {\rm d}\sigma^{VS,1}_{{\rm NNLO},N^2} = N^2 \\
&\times\Bigg\{\Bigg[\frac{1}{2}\left(\mathcal{D}_4^0(s_{13}) + \mathcal{D}_4^0(s_{23})\right) - \frac{1}{8}\left(\mathcal{D}_3^0(s_{13}) - \mathcal{D}_3^0(s_{23})\right)^2 - \frac{1}{2}\left(\tilde{\mathcal{A}}_4^0(s_{12}) - \mathcal{A}_3^0(s_{12})\,\mathcal{A}_3^0(s_{12})\right) \\
&+\frac{1}{2}\,\mathcal{D}_3^1(s_{13}) + \frac{1}{2}\,\mathcal{D}_3^1(s_{23}) - \tilde{\mathcal{A}}_3^1(s_{12})\Bigg]\, A_3^0(1_q, 3_g, 2_{\bar{q}}) \\
&+\frac{1}{2}\left(\mathcal{D}_3^0(s_{13}) + \mathcal{D}_3^0(s_{23})\right)\, A_3^1(1_q, 3_g, 2_{\bar{q}}) \\
&+\frac{b_0}{2\epsilon}\left[\mathcal{D}_3^0(s_{13})\left((s_{13})^{-\epsilon} - (s_{123})^{-\epsilon}\right) + \mathcal{D}_3^0(s_{23})\left((s_{23})^{-\epsilon} - (s_{123})^{-\epsilon}\right)\right] A_3^0(1_q, 3_g, 2_{\bar{q}})\Bigg\}{\rm d}\sigma_3\,,
\end{aligned}
\tag{6.5}
$$

where we defined the three-parton normalisation factor

$$
{\rm d}\sigma_3 = N_3\,\left(\frac{\alpha_s}{2\pi}\right)^2\,{\rm d}\Phi_3(p_1, p_2, p_3; q)\, J_3^{(3)}(p_1, p_2, p_3)\,. \tag{6.6}
$$

Combining the infrared poles of this expression with the two loop matrix element, we obtain the cancellation of all infrared poles in this colour factor,

$$
\mathcal{P}oles\left({\rm d}\sigma^{S}_{{\rm NNLO},N^2}\right) + \mathcal{P}oles\left({\rm d}\sigma^{VS,1}_{{\rm NNLO},N^2}\right) + \mathcal{P}oles\left({\rm d}\sigma^{V,2}_{{\rm NNLO},N^2}\right) = 0\,. \tag{6.7}
$$

7. Construction of the N^0 colour factor

The contribution for the N^0 colour factor to three-jet final states is more involved than all other colour factors. It receives contributions from all partonic subprocesses: $\gamma^* \to q\bar{q}ggg$ and $\gamma^* \to q\bar{q}q\bar{q}g$ at tree-level, $\gamma^* \to q\bar{q}gg$ and $\gamma^* \to q\bar{q}q\bar{q}$ at one loop and $\gamma^* \to q\bar{q}g$ at two loops. All contributions contain a mixture of colour ordered and non-ordered emissions.

7.1 Five-parton contribution

The subleading colour N^0 contribution of five-parton final states to three jet final states is

$$
\begin{aligned}
{\rm d}\sigma^{R}_{{\rm NNLO},N^0} &= N_5\,{\rm d}\Phi_5(p_1,\ldots,p_5;q) \times \Bigg[\frac{1}{3!}\Bigg(\bar{A}_5^0(1_q, 3_g, 4_g, 5_g, 2_{\bar{q}}) \\
&\quad - \sum_{(i,j,k)\in P(3,4,5)} \tilde{A}_5^0(1_q, i_g, j_g, k_g, 2_{\bar{q}})\Bigg) + \tilde{C}_5^0(1_q, 3_q, 4_{\bar{q}}, 5_g, 2_{\bar{q}}) \\
&\quad + \tilde{C}_5^0(2_{\bar{q}}, 4_{\bar{q}}, 3_q, 5_g, 1_q) - C_5^0(1_q, 3_q, 4_{\bar{q}}, 5_g, 2_{\bar{q}})\Bigg] J_3^{(5)}(p_1,\ldots,p_5) \\
&= N_5\,{\rm d}\Phi_5(p_1,\ldots,p_5;q) \times \Bigg[\frac{1}{3!}\,\bar{A}_5^0(1_q, 3_g, 4_g, 5_g, 2_{\bar{q}})
\end{aligned}
$$

$$-\frac{1}{2}\sum_{(i,j)\in P(3,4)} \tilde{A}_5^0(1_q,i_g,j_g,5_g,2_{\bar{q}})+\tilde{C}_5^0(1_q,3_q,4_{\bar{q}},5_g,2_{\bar{q}})$$

$$\left.+\tilde{C}_5^0(2_{\bar{q}},4_{\bar{q}},3_q,5_g,1_q)-C_5^0(1_q,3_q,4_{\bar{q}},5_g,2_{\bar{q}})\right] J_3^{(5)}(p_1,\ldots,p_5)\,, \quad (7.1)$$

where the symmetry factor in front of $\tilde{A}_5^0$ is due to the inherent indistinguishability of gluons. In $\tilde{A}_5^0$, gluon (5_g) is effectively photon-like. It does not participate in any three-gluon or four-gluon vertices, and there are no simple collinear limits as $(i)_g||(5)_g$ and $(j)_g||(5)_g$.

The real radiation subtraction term for this colour factor is:

$$\begin{aligned}
&\mathrm{d}\sigma^S_{\mathrm{NNLO},N^0} = N_5\,N^0\,\mathrm{d}\Phi_5(p_1,\ldots,p_5;q)\\
&\times\Bigg\{-\frac{1}{2}\sum_{(i,j)\in(3,4)}\Big[A_3^0(1_q,5_g,2_{\bar{q}})\,A_4^0(\widetilde{(15)}_q,i_g,j_g,\widetilde{(25)}_{\bar{q}})J_3^{(4)}(\widetilde{p_{15}},p_i,p_j,\widetilde{p_{25}})\\
&+d_3^0(1,i,j)\,\tilde{A}_4^0(\widetilde{(1i)}_q,\widetilde{(ji)}_g,5_g,2_{\bar{q}})J_3^{(4)}(\widetilde{p_{1i}},\widetilde{p_{ji}},p_5,p_2)\\
&+d_3^0(2,j,i)\,\tilde{A}_4^0(1_q,5_g,\widetilde{(ij)}_g,\widetilde{(2j)}_{\bar{q}})J_3^{(4)}(p_1,p_5,\widetilde{p_{ij}},\widetilde{p_{2j}})\Big]\\
&+\frac{1}{3!}\sum_{(i,j,k)\in P_C(3,4,5)} A_3^0(1_q,i_g,2_{\bar{q}})\,\tilde{A}_4^0(\widetilde{(1i)}_q,j_g,k_g,\widetilde{(2i)}_{\bar{q}})J_3^{(4)}(\widetilde{p_{1i}},p_j,p_k,\widetilde{p_{2i}})\\
&-A_3^0(1_q,5_g,3_q)\,\Big[C_4^0(\widetilde{(15)}_q,\widetilde{(35)}_q,4_{\bar{q}},2_{\bar{q}})+C_4^0(2_{\bar{q}},4_{\bar{q}},\widetilde{(35)}_q,\widetilde{(15)}_q)\Big]\,J_3^{(4)}(\widetilde{p_{15}},\widetilde{p_{35}},p_4,p_2)\\
&-A_3^0(2_{\bar{q}},5_g,4_{\bar{q}})\,\Big[C_4^0(1_q,3_q,\widetilde{(45)}_{\bar{q}},\widetilde{(25)}_{\bar{q}})+C_4^0(\widetilde{(25)}_{\bar{q}},\widetilde{(45)}_{\bar{q}},3_q,1_q)\Big]\,J_3^{(4)}(p_1,p_3,\widetilde{p_{25}},\widetilde{p_{45}})\\
&-\frac{1}{2}\sum_{(i,j)\in(3,4)}\Bigg(A_4^0(1_q,i_g,j_g,2_{\bar{q}})-d_3^0(1_q,i_g,j_g)\,A_3^0(\widetilde{(1i)}_q,\widetilde{(ji)}_g,2_{\bar{q}})\\
&-d_3^0(2_{\bar{q}},j_g,i_g)\,A_3^0(1_q,\widetilde{(ij)}_g,\widetilde{(2j)}_{\bar{q}})\Bigg)\,A_3^0(\widetilde{(1ij)}_q,5_g,\widetilde{(2ji)}_{\bar{q}})\,J_3^{(3)}(\widetilde{p_{1ij}},p_5,\widetilde{p_{2ji}})\\
&-\frac{1}{3}\sum_{(i,j)\in(3,4)}\Bigg(\tilde{A}_4^0(1_q,i_g,5_g,2_{\bar{q}})-A_3^0(1_q,i_g,2_{\bar{q}})\,A_3^0(\widetilde{(1i)}_q,5_g,\widetilde{(2i)}_{\bar{q}})\\
&-A_3^0(1_q,5_g,2_{\bar{q}})\,A_3^0(\widetilde{(15)}_q,i_g,\widetilde{(25)}_{\bar{q}})\Bigg)\,A_3^0(\widetilde{(1i5)}_q,j_g,\widetilde{(2i5)}_{\bar{q}})\,J_3^{(3)}(\widetilde{p_{1i5}},p_j,\widetilde{p_{2i5}})\\
&+\frac{1}{6}\Bigg(\tilde{A}_4^0(1_q,3_g,4_g,2_{\bar{q}})-A_3^0(1_q,3_g,2_{\bar{q}})\,A_3^0(\widetilde{(13)}_q,4_g,\widetilde{(23)}_{\bar{q}})\\
&-A_3^0(1_q,4_g,2_{\bar{q}})\,A_3^0(\widetilde{(14)}_q,3_g,\widetilde{(24)}_{\bar{q}})\Bigg)\,A_3^0(\widetilde{(134)}_q,5_g,\widetilde{(234)}_{\bar{q}})\,J_3^{(3)}(\widetilde{p_{134}},p_5,\widetilde{p_{234}})\\
&-\Big[C_4^0(1_q,3_q,4_{\bar{q}},2_{\bar{q}})+C_4^0(2_{\bar{q}},4_{\bar{q}},3_q,1_q)\Big]\,A_3^0(\widetilde{(134)}_q,5_g,\widetilde{(234)}_{\bar{q}})\,J_3^{(3)}(\widetilde{p_{134}},p_5,\widetilde{p_{234}})\\
&+\frac{1}{2}\sum_{(i,j)\in(3,4)}\Bigg(d_3^0(1_q,i_g,j_g)\,A_3^0(\widetilde{(1i)}_q,5_g,2_{\bar{q}})\,A_3^0(\widetilde{(\widetilde{(1i)}5)}_q,\widetilde{(ji)}_g,\widetilde{(25)}_{\bar{q}})\,J_3^{(3)}(\widetilde{p_{(1i)5}},\widetilde{p_{ji}},\widetilde{p_{25}})
\end{aligned}$$

$$+d_3^0(2_{\bar q},j_g,i_g)\,A_3^0(1_q,5_g,\widetilde{(2j)}_{\bar q})\,A_3^0(\widetilde{(15)}_q,\widetilde{(ij)}_g,\widetilde{((2j)5)}_{\bar q})\,J_3^{(3)}(\widetilde{p_{15}},\widetilde{p_{ij}},\widetilde{p_{(2j)5}})$$
$$-A_3^0(1_q,i_g,2_{\bar q})\,A_3^0(\widetilde{(1i)}_q,5_g,\widetilde{(2i)}_{\bar q})\,A_3^0(\widetilde{((1i)5)}_q,j_g,\widetilde{((2i)5)}_{\bar q})\,J_3^{(3)}(\widetilde{p_{(1i)5}},\widetilde{p_{(2i)5}},p_j)\Bigg)\Bigg\}. \quad (7.2)$$

7.2 Four-parton contribution

The four-parton contribution comes from the subleading colour one-loop correction to $\gamma^* \to q\bar{q}gg$, where the gluonic emissions are colour-ordered, from the leading colour one-loop correction to $\gamma^* \to q\bar{q}gg$, where the gluonic emissions are not colour-ordered and from the one-loop correction to the identical-flavour process $\gamma^* \to q\bar{q}q\bar{q}$. It reads:

$$\begin{aligned}
\mathrm{d}\sigma^{V,1}_{\mathrm{NNLO},N^0} &= N_4\,N^0\,\left(\frac{\alpha_s}{2\pi}\right)\,\mathrm{d}\Phi_4(p_1,\ldots,p_4;q)\\
&\qquad\times\Bigg\{-\frac{1}{2}\Bigg(\sum_{(i,j)\in(3,4)}A_4^{1,b}(1_q,i_g,j_g,2_{\bar q})-\tilde{A}_4^{1,a}(1_q,3_g,4_g,2_{\bar q})+\tilde{A}_4^{1,d}(1_q,3_g,4_g,2_{\bar q})\Bigg)\\
&\qquad-C_4^{1,d}(1_q,3_q,4_{\bar q},2_{\bar q})-C_4^{1,d}(2_{\bar q},4_{\bar q},3_q,1_q)\Bigg\}\,J_3^{(4)}(p_1,p_2,p_3,p_4), \qquad (7.3)
\end{aligned}$$

The one-loop single unresolved subtraction term for this colour factor is

$$\begin{aligned}
&\mathrm{d}\sigma^{VS,1}_{\mathrm{NNLO},N^0} = N_4\,N^0\,\left(\frac{\alpha_s}{2\pi}\right)\,\mathrm{d}\Phi_4(p_1,p_2,p_3,p_4;q)\\
&\Bigg\{\frac{1}{2}\sum_{(i,j)\in(3,4)}\Bigg[\left(\frac{1}{2}\mathcal{D}_3^0(s_{1i})+\frac{1}{2}\mathcal{D}_3^0(s_{2j})\right)\,\tilde{A}_4^0(1_q,i_g,j_g,2_{\bar q})\\
&+\mathcal{A}_3^0(s_{12})A_4^0(1_q,i_g,j_g,2_{\bar q})\Bigg]J_3^{(4)}(p_1,p_2,p_3,p_4)\\
&-\frac{1}{2}\,\mathcal{A}_3^0(s_{12})\tilde{A}_4^0(1_q,3_g,4_g,2_{\bar q})J_3^{(4)}(p_1,p_2,p_3,p_4)\\
&+\left(\mathcal{A}_3^0(s_{13})+\mathcal{A}_3^0(s_{24})\right)\left(C_4^0(1_q,3_q,4_{\bar q},2_{\bar q})+C_4^0(2_{\bar q},4_{\bar q},3_q,1_q)\right)\,J_3^{(4)}(p_1,p_2,p_3,p_4)\\
&+\frac{1}{2}\sum_{(i,j)\in(3,4)}\Bigg\{-\Bigg[d_3^0(1_q,i_g,j_g)\,\Big[\tilde{A}_3^1(\widetilde{(1i)}_q,\widetilde{(ji)}_g,2_{\bar q})+\mathcal{A}_2^1(s_{1234})A_3^0(\widetilde{(1i)}_q,\widetilde{(ji)}_g,2_{\bar q})\Big]\\
&J_3^{(3)}(\widetilde{p_{1i}},\widetilde{p_{ji}},p_2)+(1\leftrightarrow 2)\Bigg]-\tilde{A}_3^1(1_q,i_g,2_{\bar q})\,A_3^0(\widetilde{(1i)}_q,j_g,\widetilde{(2i)}_{\bar q})\,J_3^{(3)}(\widetilde{p_{1i}},p_j,\widetilde{p_{2i}})\\
&-A_3^0(1_q,i_g,2_{\bar q})\,\Big[A_3^1(\widetilde{(1i)}_q,j_g,\widetilde{(2i)}_{\bar q})+\mathcal{A}_2^1(s_{1234})A_3^0(\widetilde{(1i)}_q,j_g,\widetilde{(2i)}_{\bar q})\Big]\,J_3^{(3)}(\widetilde{p_{1i}},p_j,\widetilde{p_{2i}})\\
&-A_3^1(1_q,i_g,2_{\bar q})\,A_3^0(\widetilde{(1i)}_q,j_g,\widetilde{(2i)}_{\bar q})\,J_3^{(3)}(\widetilde{p_{1i}},p_j,\widetilde{p_{2i}})\\
&-\Big[d_3^0(1_q,i_g,j_g)\,\mathcal{A}_3^0(s_{\widetilde{(1i)}2})\,A_3^0(\widetilde{(1i)}_q,\widetilde{(ji)}_g,2_{\bar q})J_3^{(3)}(\widetilde{p_{1i}},\widetilde{p_{ji}},p_2)+(1\leftrightarrow 2)\Big]\\
&-\left[\mathcal{A}_3^0(s_{12})-\mathcal{A}_3^0(s_{12i})\right]\,A_3^0(1_q,i_g,2_{\bar q})\,A_3^0(\widetilde{(1i)}_q,j_g,\widetilde{(2i)}_{\bar q})\,J_3^{(3)}(\widetilde{p_{1i}},p_j,\widetilde{p_{2i}})\\
&-\frac{1}{2}\,\Big[\mathcal{D}_3^0(s_{\widetilde{(1i)}j})+\mathcal{D}_3^0(s_{\widetilde{(2i)}j})\Big]\,A_3^0(1_q,i_g,2_{\bar q})\,A_3^0(\widetilde{(1i)}_q,j_g,\widetilde{(2i)}_{\bar q})\,J_3^{(3)}(\widetilde{p_{1i}},p_j,\widetilde{p_{2i}})\\
&-\Big[\frac{1}{2}\mathcal{D}_3^0(s_{1i})+\frac{1}{2}\mathcal{D}_3^0(s_{2i})-\mathcal{A}_3^0(s_{12i})\Big]\,A_3^0(1_q,i_g,2_{\bar q})\,A_3^0(\widetilde{(1i)}_q,j_g,\widetilde{(2i)}_{\bar q})\,J_3^{(3)}(\widetilde{p_{1i}},p_j,\widetilde{p_{2i}})\\
&+\Big[\frac{1}{2}\mathcal{D}_3^0(s_{\widetilde{(1i)}j})+\frac{1}{2}\mathcal{D}_3^0(s_{\widetilde{(2i)}j})-\mathcal{A}_3^0(s_{12i})-\frac{1}{2}\mathcal{D}_3^0(s_{1j})-\frac{1}{2}\mathcal{D}_3^0(s_{2j})+\mathcal{A}_3^0(s_{12})\Big]
\end{aligned}$$

$$
\begin{aligned}
&A_3^0(1_q,i_g,2_{\bar q})\,A_3^0(\widetilde{(1i)}_q,j_g,\widetilde{(2i)}_{\bar q})\,J_3^{(3)}(\widetilde{p_{1i}},p_j,\widetilde{p_{2i}})\\
&-b_0\log\frac{q^2}{s_{12i}}A_3^0(1_q,i_g,2_{\bar q})\,A_3^0(\widetilde{(1i)}_q,j_g,\widetilde{(2i)}_{\bar q})J_3^{(3)}(\widetilde{p_{1i}},p_j,\widetilde{p_{2i}})\Bigg\}
\end{aligned}
\tag{7.4}
$$

7.3 Three-parton contribution

The three parton contribution to the N^0 colour factor receives contributions from the three-parton virtual two-loop correction and the integrated five-parton tree-level and four-parton one-loop subtraction terms, which read

$$
\begin{aligned}
\mathrm{d}\sigma^S_{\mathrm{NNLO},N^0}+\mathrm{d}\sigma^{VS,1}_{\mathrm{NNLO},N^0} = &\; N^0\\
&\times\Bigg\{-\Bigg[\mathcal{A}_4^0(s_{12})+\frac{1}{2}\tilde{\mathcal{A}}_4^0(s_{12})+2\,\mathcal{C}_4^0(s_{12})+\frac{1}{2}\mathcal{A}_3^0(s_{12})\left(\mathcal{D}_3^0(s_{13})+\mathcal{D}_3^0(s_{23})\right)\\
&-\mathcal{A}_3^0(s_{12})\,\mathcal{A}_3^0(s_{12})+\mathcal{A}_3^1(s_{12})+\tilde{\mathcal{A}}_3^1(s_{12})\Bigg]\,A_3^0(1_q,3_g,2_{\bar q})\\
&-\frac{1}{2}\left(\mathcal{D}_3^0(s_{13})+\mathcal{D}_3^0(s_{23})\right)\,\tilde{A}_3^1(1_q,3_g,2_{\bar q})-\mathcal{A}_3^0(s_{13})\,A_3^1(1_q,3_g,2_{\bar q})\\
&-\frac{b_0}{\epsilon}\,\mathcal{A}_3^0(s_{12})\left((s_{12})^{-\epsilon}-(s_{123})^{-\epsilon}\right)\,A_3^0(1_q,3_g,2_{\bar q})\Bigg\}\mathrm{d}\sigma_3\,.
\end{aligned}
\tag{7.5}
$$

Combining the infrared poles of this expression with the two loop matrix element, we obtain the cancellation of all infrared poles in this colour factor,

$$
\mathcal{P}oles\left(\mathrm{d}\sigma^S_{\mathrm{NNLO},N^0}\right)+\mathcal{P}oles\left(\mathrm{d}\sigma^{VS,1}_{\mathrm{NNLO},N^0}\right)+\mathcal{P}oles\left(\mathrm{d}\sigma^{V,2}_{\mathrm{NNLO},N^0}\right)=0\,. \tag{7.6}
$$

8. Construction of the $1/N^2$ colour factor

The $1/N^2$ colour factor receives contributions from five-parton tree-level $\gamma^*\to q\bar q ggg$ and $\gamma^*\to q\bar q q\bar q g$, four-parton one-loop $\gamma^*\to q\bar q gg$ and $\gamma^*\to q\bar q q\bar q$ as well as tree-level two-loop $\gamma^*\to q\bar q g$. The gluon emissions are all photon-like, not containing any gluon self-coupling. The four-quark processes contribute through the identical-quark-only terms.

The construction of the subtraction terms for this colour factor was discussed in detail in [65, 32].

8.1 Five-parton contribution

Two different five-parton final states contribute at $1/N^2$ to three-jet final states at NNLO: $\gamma^*\to q\bar q ggg$ and $\gamma^*\to q\bar q q\bar q g$ with identical quarks.

The NNLO radiation term appropriate for the three jet final state is given by

$$
\begin{aligned}
\mathrm{d}\sigma^R_{\mathrm{NNLO},1/N^2} = &\;\frac{N_5}{N^2}\,\mathrm{d}\Phi_5(p_1,\ldots,p_5;q) \qquad (8.1)\\
&\times\left[\frac{1}{3!}\,\bar{A}_5^0(1_q,3_g,4_g,5_g,2_{\bar q})+2\,\tilde{C}_5^0(1_q,2_{\bar q},3_q,4_{\bar q},5_g)\right]J_3^{(5)}(p_1,\ldots,p_5)\,,
\end{aligned}
$$

where the symmetry factor in front of $\bar{A}_5^0$ is due to the inherent indistinguishability of gluons. The factor 2 in front of $\tilde{C}_5^0$ arises from the fact that two different momentum arrangements contribute to the squared matrix element (4.21). If the quarks and antiquarks are not distinguished by the jet functions, these contribute equally.

The real radiation subtraction term for this colour factor is:

$$
\begin{aligned}
\mathrm{d}\sigma^{S}_{\mathrm{NNLO},1/N^2} &= \frac{N_5}{N^2}\,\mathrm{d}\Phi_5(p_1,\ldots,p_5;q) \\
&\Bigg\{\frac{1}{3!}\sum_{i,j,k\in P_C(3,4,5)}\Bigg[A_3^0(1_q,i_g,2_{\bar q})\,\tilde{A}_4^0(\widetilde{(1i)}_q,j_g,k_g,\widetilde{(2i)}_{\bar q})\;J_3^{(4)}(\widetilde{p_{1i}},p_j,p_k,\widetilde{p_{2i}}) \\
&+\Bigg(\tilde{A}_4^0(1_q,i_g,j_g,2_{\bar q}) - A_3^0(1_q,i_g,2_{\bar q})\;A_3^0(\widetilde{(1i)}_q,j_g,\widetilde{(2i)}_{\bar q}) \\
&-A_3^0(1_q,j_g,2_{\bar q})\;A_3^0(\widetilde{(1j)}_q,i_g,\widetilde{(2j)}_{\bar q})\;\Bigg)A_3^0(\widetilde{(1ij)}_q,k_g,\widetilde{(2ij)}_{\bar q})\;J_3^{(3)}(\widetilde{p_{1ij}},p_k,\widetilde{p_{2ij}})\Bigg] \\
&+2\Bigg[A_3^0(1_q,5_g,2_{\bar q})\;C_4^0(\widetilde{(15)}_q,3_q,4_{\bar q},\widetilde{(25)}_{\bar q})\;J_3^{(4)}(\widetilde{p_{15}},p_3,p_4,\widetilde{p_{25}}) \\
&+A_3^0(1_q,5_g,4_{\bar q})\;C_4^0(\widetilde{(15)}_q,3_q,\widetilde{(45)}_{\bar q},2_{\bar q})\;J_3^{(4)}(\widetilde{p_{15}},p_2,p_3,\widetilde{p_{45}}) \\
&+A_3^0(3_q,5_g,2_{\bar q})\;C_4^0(1_q,\widetilde{(35)}_q,4_{\bar q},\widetilde{(25)}_{\bar q})\;J_3^{(4)}(\widetilde{p_{35}},p_1,p_4,\widetilde{p_{25}}) \\
&+A_3^0(3_q,5_g,4_{\bar q})\;C_4^0(1_q,\widetilde{(35)}_q,\widetilde{(45)}_{\bar q},2_{\bar q})\;J_3^{(4)}(\widetilde{p_{35}},p_1,p_2,\widetilde{p_{45}}) \\
&-A_3^0(1_q,5_g,3_q)\;C_4^0(\widetilde{(15)}_q,\widetilde{(35)}_q,4_{\bar q},2_{\bar q})\;J_3^{(4)}(\widetilde{p_{15}},p_2,p_4,\widetilde{p_{35}}) \\
&-A_3^0(2_{\bar q},5_g,4_{\bar q})\;C_4^0(1_q,3_q,\widetilde{(45)}_{\bar q},\widetilde{(25)}_{\bar q})\;J_3^{(4)}(\widetilde{p_{25}},p_1,p_3,\widetilde{p_{45}}) \\
&+C_4^0(1_q,3_q,4_{\bar q},2_{\bar q})\;A_3^0(\widetilde{(134)}_q,5_g,\widetilde{(234)}_{\bar q})\;J_3^{(3)}(\widetilde{p_{134}},p_5,\widetilde{p_{234}})\Bigg]\Bigg\}\,. \qquad (8.2)
\end{aligned}
$$

The sum in the first contribution runs only over the three cyclic permutations of the gluon momenta to prevent double counting of identical configurations obtained by interchange of j and k.

8.2 Four-parton contribution

At one-loop, there are two contributions to the colour suppressed contribution proportional to $1/N^2$, one from the four quark final state and one from the two quark-two gluon final state:

$$
\begin{aligned}
\mathrm{d}\sigma^{V,1}_{\mathrm{NNLO},1/N^2} &= \frac{N_4}{N^2}\left(\frac{\alpha_s}{2\pi}\right)\mathrm{d}\Phi_4(p_1,\ldots,p_4;q) \\
&\left\{\frac{1}{2!}\,\tilde{A}_4^{1,b}(1_q,3_g,4_g,2_{\bar q}) + 2\,C_4^{1,e}(1_q,3_q,4_{\bar q},2_{\bar q})\right\}\,J_3^{(4)}(p_1,\ldots,p_4), \qquad (8.3)
\end{aligned}
$$

where the origin of the symmetry factors is as in the real radiation five-parton contributions of the previous section.

The corresponding subtraction term is:

$$
\begin{aligned}
\mathrm{d}\sigma^{VS,1}_{\mathrm{NNLO},1/N^2} &= \frac{N_4}{N^2}\left(\frac{\alpha_s}{2\pi}\right)\mathrm{d}\Phi_4(p_1,\ldots,p_4;q) \\
&\times\Bigg\{\frac{1}{2!}\sum_{i,j\in P(3,4)}\Bigg[-\mathcal{A}_3^0(s_{12})\tilde{A}_4^0(1_q,i_g,j_g,2_{\bar q})J_3^{(4)}(p_1,p_i,p_j,p_2)
\end{aligned}
$$

$$
\begin{aligned}
&+\Bigg(A_3^0(1_q,i_g,2_{\bar{q}})\left[\tilde{A}_3^1(\widetilde{(1i)}_q,j_g,\widetilde{(2i)}_{\bar{q}})+\mathcal{A}_2^1(s_{1234})A_3^0(\widetilde{(1i)}_q,j_g,\widetilde{(2i)}_{\bar{q}})\right]\\
&+\tilde{A}_3^1(1_q,i_g,2_{\bar{q}})A_3^0(\widetilde{(1i)}_q,j_g,\widetilde{(2i)}_{\bar{q}})\Bigg)J_3^{(3)}(\widetilde{p_{1i}},p_j,\widetilde{p_{2i}})\\
&+\mathcal{A}_3^0(s_{12})A_3^0(1_q,i_g,2_{\bar{q}})A_3^0(\widetilde{(1i)}_q,j_g,\widetilde{(2i)}_{\bar{q}})J_3^{(3)}(\widetilde{p_{1i}},p_j,\widetilde{p_{2i}})\Bigg]\\
&-2\Big[\mathcal{A}_3^0(s_{12})+\mathcal{A}_3^0(s_{14})+\mathcal{A}_3^0(s_{23})+\mathcal{A}_3^0(s_{34})-\mathcal{A}_3^0(s_{13})-\mathcal{A}_3^0(s_{24})\Big]\\
&\times C_4^0(1_q,3_q,4_{\bar{q}},2_{\bar{q}})J_3^{(4)}(p_1,p_3,p_4,p_2)\Bigg\}\,. \qquad (8.4)
\end{aligned}
$$

8.3 Three-parton contribution

The three-parton contribution consists of the two-loop three-parton matrix element together with the integrated forms of the five-parton and four-parton subtraction terms,

$$
\begin{aligned}
\mathrm{d}\sigma^{S}_{\mathrm{NNLO},1/N^2}+\mathrm{d}\sigma^{VS,1}_{\mathrm{NNLO},1/N^2} = \frac{1}{N^2}\Bigg\{&\left[\frac{1}{2}\tilde{\mathcal{A}}_4^0(s_{12})+2\,\mathcal{C}_4^0(s_{12})+\tilde{\mathcal{A}}_3^1(s_{12})\right]A_3^0(1_q,3_g,2_{\bar{q}})\\
&+\mathcal{A}_3^0(s_{13})\,\tilde{A}_3^1(1_q,3_g,2_{\bar{q}})\Bigg\}\mathrm{d}\sigma_3\,. \qquad (8.5)
\end{aligned}
$$

Combining the infrared poles of this expression with the two loop matrix element, we obtain the cancellation of all infrared poles in this colour factor,

$$
\mathcal{P}oles\left(\mathrm{d}\sigma^{S}_{\mathrm{NNLO},1/N^2}\right)+\mathcal{P}oles\left(\mathrm{d}\sigma^{VS,1}_{\mathrm{NNLO},1/N^2}\right)+\mathcal{P}oles\left(\mathrm{d}\sigma^{V,2}_{\mathrm{NNLO},1/N^2}\right)=0\,. \qquad (8.6)
$$

9. Construction of the $N_F\,N$ colour factor

The colour factor $N_F\,N$ receives contribution from the colour-ordered five-parton tree-level process $\gamma^*\to q\bar{q}q'\bar{q}'g$, the four-parton one-loop processes $\gamma^*\to q\bar{q}q'\bar{q}'$ and $\gamma^*\to q\bar{q}gg$ at leading colour and from the two-loop three-parton process $\gamma^*\to q\bar{q}g$.

9.1 Five-parton contribution

The NNLO radiation term appropriate for the three jet final state is given by

$$
\begin{aligned}
\mathrm{d}\sigma^{R}_{\mathrm{NNLO},N_FN} &= N_5\,N_FN\,\mathrm{d}\Phi_5(p_1,\ldots,p_5;q) \qquad (9.1)\\
&\times\left[B_5^{0,a}(1_q,5_g,4_{\bar{q}'};3_{q'},2_{\bar{q}})+B_5^{0,b}(1_q,4_{\bar{q}'};3_{q'},5_g,2_{\bar{q}})\right]J_3^{(5)}(p_1,\ldots,p_5),
\end{aligned}
$$

The two terms represent the two colour orderings of the leading colour amplitude for this process. Since the leading colour $qq'\bar{q}'g$-antenna subtraction terms allows q to represent either a quark or an antiquark, both colour orderings are mixed together. Therefore, it is not possible to construct a subtraction term for an individual contribution, but only for their sum.

The subtraction term for this contribution is

$$
\begin{aligned}
\mathrm{d}\sigma^{S}_{\mathrm{NNLO},N_FN} &= N_5\, N_F N\, \mathrm{d}\Phi_5(p_1,\dots,p_5;q) \qquad (9.2)\\
&\times\Big\{ A_3^0(1_q,5_g,4_{\bar q'})\, B_4^0(\widetilde{(15)}_q,3_{q'},\widetilde{(45)}_{\bar q'},2_{\bar q})\, J_3^{(4)}(\widetilde{p_{15}},p_2,p_3,\widetilde{p_{45}})\\
&+A_3^0(3_{q'},5_g,2_{\bar q})\, B_4^0(1_q,\widetilde{(35)}_{q'},4_{\bar q'},\widetilde{(25)}_{\bar q})\, J_3^{(4)}(p_1,\widetilde{p_{25}},\widetilde{p_{35}},p_4)\\
&+G_3^0(5_g,3_{q'},4_{\bar q'})A_4^0(1_q,\widetilde{(34)}_g,\widetilde{(54)}_g,2_{\bar q})\, J_3^{(4)}(p_1,p_2,\widetilde{p_{34}},\widetilde{p_{54}})\\
&+G_3^0(5_g,3_{q'},4_{\bar q'})A_4^0(1_q,\widetilde{(54)}_g,\widetilde{(34)}_g,2_{\bar q})\, J_3^{(4)}(p_1,p_2,\widetilde{p_{34}},\widetilde{p_{54}})\\
&+\Big(E^0_{4,a}(1_q,3_{q'},4_{\bar q'},5_g)\\
&-G_3^0(5_g,3_{q'},4_{\bar q'})\, d_3^0(1_q,\widetilde{(34)}_g,\widetilde{(54)}_g)\Big)\, A_3^0(\widetilde{(134)}_q,\widetilde{(543)}_g,2_{\bar q})J_3^{(3)}(\widetilde{p_{134}},p_2,\widetilde{p_{543}})\\
&+\Big(E^0_{4,b}(1_q,3_{q'},4_{\bar q'},5_g) - A_3^0(1_q,5_g,4_{\bar q'})\, E_3^0(\widetilde{(15)}_q,3_{q'},\widetilde{(45)}_{\bar q'})\\
&-G_3^0(5_g,3_{q'},4_{\bar q'})\, d_3^0(1_q,\widetilde{(54)}_g,\widetilde{(34)}_g)\Big)\, A_3^0(\widetilde{(154)}_q,\widetilde{(345)}_g,2_{\bar q})J_3^{(3)}(\widetilde{p_{154}},p_2,\widetilde{p_{345}})\\
&+\Big(E^0_{4,a}(2_{\bar q},4_{\bar q'},3_{q'},5_g)\\
&-G_3^0(5_g,3_{q'},4_{\bar q'})\, d_3^0(2_{\bar q},\widetilde{(43)}_g,\widetilde{(53)}_g)\Big)\, A_3^0(1_q,\widetilde{(534)}_g,\widetilde{(243)}_{\bar q})J_3^{(3)}(p_1,\widetilde{p_{243}},\widetilde{p_{534}})\\
&+\Big(E^0_{4,b}(2_{\bar q},4_{\bar q'},3_{q'},5_g) - A_3^0(3_{q'},5_g,2_{\bar q})\, E_3^0(\widetilde{(25)}_{\bar q},4_{\bar q'},\widetilde{(35)}_{q'})\\
&-G_3^0(5_g,3_{q'},4_{\bar q'})\, d_3^0(2_{\bar q},\widetilde{(53)}_g,\widetilde{(43)}_g)\Big)\, A_3^0(1_q,\widetilde{(435)}_g,\widetilde{(253)}_{\bar q})J_3^{(3)}(p_1,\widetilde{p_{253}},\widetilde{p_{435}})\Big\}\,.
\end{aligned}
$$

9.2 Four-parton contribution

The four parton contribution to the N_FN colour factor reads:

$$
\begin{aligned}
\mathrm{d}\sigma^{V,1}_{\mathrm{NNLO},N_FN} &= N_4\, N_F N\, \left(\frac{\alpha_s}{2\pi}\right)\, \mathrm{d}\Phi_4(p_1,\dots,p_4;q) \qquad (9.3)\\
&\left\{ \frac{1}{2}\sum_{(i,j)\in(3,4)} \Big(B_4^{1,a}(1_q,i_{q'},j_{\bar q'},2_{\bar q}) + A_4^{1,c}(1_q,i_g,j_g,2_{\bar q})\Big)\right\} J_3^{(4)}(p_1,\dots,p_4)\,.
\end{aligned}
$$

The average over the permutations of the momenta (3) and (4) has to be made in both contributions to this colour factor. In the one-loop correction to the $\gamma^* \to qq'\bar q'\bar q$ final state $B_4^{1,a}$, the secondary quark-antiquark pair has to be symmetrised, since the quark-gluon antenna functions used in the one-loop subtraction terms do not distinguish quarks and antiquarks. The summation over the two colour orderings of the one-loop correction to the $\gamma^* \to qgg\bar q$ final state $A_4^{1,c}$ must be kept since the one-loop subtraction functions appropriate to this term contain both orderings because of their cyclicity.

The corresponding subtraction term is:

$$
\begin{aligned}
\mathrm{d}\sigma^{VS,1}_{\mathrm{NNLO},N_FN} &= N_4 N_F N \left(\frac{\alpha_s}{2\pi}\right) \mathrm{d}\Phi_4(p_1,\ldots,p_4;q) \\
&\times\Bigg\{ -\frac{1}{2}\sum_{(i,j)\in(3,4)} \left[\mathcal{A}_3^0(s_{1j})+\mathcal{A}_3^0(s_{2i})\right]\, B_4^0(1_q,i_{q'},j_{\bar{q}'},2_{\bar{q}})\, J_3^{(4)}(p_1,\ldots,p_4) \\
&-\sum_{(i,j)\in(3,4)} \mathcal{G}_3^0(s_{ij})\, A_4^0(1_q,i_g,j_g,2_{\bar{q}})\, J_3^{(4)}(p_1,\ldots,p_4) \\
&+\Bigg\{\frac{1}{2}\Bigg(E_3^0(1_q,3_{q'},4_{\bar{q}'})\left[A_3^1(\widetilde{(13)}_q,\widetilde{(43)}_g,2_{\bar{q}})+\mathcal{A}_2^1(s_{1234})A_3^0(\widetilde{(13)}_q,\widetilde{(43)}_g,2_{\bar{q}})\right] \\
&+E_3^1(1_q,3_{q'},4_{\bar{q}'})\, A_3^0(\widetilde{(13)}_q,\widetilde{(43)}_g,2_{\bar{q}}) \\
&+\frac{1}{2}\left(\mathcal{D}_3^0(s_{134})+\mathcal{D}_3^0(s_{2\widetilde{(43)}})\right)\, E_3^0(1_q,3_{q'},4_{\bar{q}'})\, A_3^0(\widetilde{(13)}_q,\widetilde{(43)}_g,2_{\bar{q}}) \\
&+\left(\mathcal{A}_3^0(s_{13})+\mathcal{A}_3^0(s_{14})-\mathcal{D}_3^0(s_{134})\right)\, E_3^0(1_q,3_{q'},4_{\bar{q}'})\, A_3^0(\widetilde{(13)}_q,\widetilde{(43)}_g,2_{\bar{q}}) \\
&+b_0\log\frac{q^2}{s_{134}}E_3^0(1_q,3_{q'},4_{\bar{q}'})\, A_3^0(\widetilde{(13)}_q,\widetilde{(43)}_g,2_{\bar{q}})\Bigg)J_3^{(3)}(\widetilde{p_{13}},\widetilde{p_{43}},p_2)+(1\leftrightarrow 2)\Bigg\} \\
&+\Bigg\{\frac{1}{2}\Bigg(D_3^0(1_q,3_g,4_g)\hat{A}_3^1(\widetilde{(13)}_q,\widetilde{(43)}_g,2_{\bar{q}})+\hat{D}_3^1(1_q,3_g,4_g)\, A_3^0(\widetilde{(13)}_q,\widetilde{(43)}_g,2_{\bar{q}}) \\
&+2\mathcal{G}_3^0(s_{34})\, D_3^0(1_q,3_g,4_g)\, A_3^0(\widetilde{(13)}_q,\widetilde{(43)}_g,2_{\bar{q}}) \\
&+b_{0,F}\log\frac{q^2}{s_{134}}D_3^0(1_q,3_g,4_g)\, A_3^0(\widetilde{(13)}_q,\widetilde{(43)}_g,2_{\bar{q}})\Bigg)J_3^{(3)}(\widetilde{p_{13}},\widetilde{p_{43}},p_2)+(1\leftrightarrow 2)\Bigg\}\,.
\end{aligned}
\tag{9.4}
$$

9.3 Three-parton contribution

The three parton contribution to the $N_F N$ colour factor contains the three-parton virtual two-loop correction and the integrated five-parton tree-level and four-parton one-loop subtraction terms, which read

$$
\begin{aligned}
&\mathrm{d}\sigma^{S}_{\mathrm{NNLO},N_FN}+\mathrm{d}\sigma^{VS,1}_{\mathrm{NNLO},N_FN} = N_F\, N \\
&\times\Bigg\{\Bigg[\mathcal{E}_4^0(s_{13})+\mathcal{E}_4^0(s_{23})-\frac{1}{4}\left(\mathcal{D}_3^0(s_{13})\,\mathcal{E}_3^0(s_{13})+\mathcal{D}_3^0(s_{23})\,\mathcal{E}_3^0(s_{23})\right) \\
&+\frac{1}{4}\left(\mathcal{D}_3^0(s_{13})\,\mathcal{E}_3^0(s_{23})+\mathcal{D}_3^0(s_{23})\,\mathcal{E}_3^0(s_{13})\right)+\frac{1}{2}\,\hat{\mathcal{D}}_3^1(s_{13})+\frac{1}{2}\,\hat{\mathcal{D}}_3^1(s_{23})+\frac{1}{2}\,\mathcal{E}_3^1(s_{13}) \\
&+\frac{1}{2}\,\mathcal{E}_3^1(s_{23})\Bigg]\, A_3^0(1_q,3_g,2_{\bar{q}})+\frac{1}{2}\left(\mathcal{E}_3^0(s_{13})+\mathcal{E}_3^0(s_{23})\right)\, A_3^1(1_q,3_g,2_{\bar{q}}) \\
&+\frac{1}{2}\left(\mathcal{D}_3^0(s_{13})+\mathcal{D}_3^0(s_{23})\right)\, \hat{A}_3^1(1_q,3_g,2_{\bar{q}}) \\
&+\frac{b_{0,F}}{2\epsilon}\left[\mathcal{D}_3^0(s_{13})\left((s_{13})^{-\epsilon}-(s_{123})^{-\epsilon}\right)+\mathcal{D}_3^0(s_{23})\left((s_{23})^{-\epsilon}-(s_{123})^{-\epsilon}\right)\right]\, A_3^0(1_q,3_g,2_{\bar{q}}) \\
&+\frac{b_0}{2\epsilon}\left[\mathcal{E}_3^0(s_{13})\left((s_{13})^{-\epsilon}-(s_{123})^{-\epsilon}\right)+\mathcal{E}_3^0(s_{23})\left((s_{23})^{-\epsilon}-(s_{123})^{-\epsilon}\right)\right]\, A_3^0(1_q,3_g,2_{\bar{q}})\Bigg\}\mathrm{d}\sigma_3\,.
\end{aligned}
\tag{9.5}
$$

Combining the infrared poles of this expression with the two loop matrix element, we obtain the cancellation of all infrared poles in this colour factor,

$$\mathcal{P}oles\left(\mathrm{d}\sigma^{S}_{\mathrm{NNLO},N_F N}\right)+\mathcal{P}oles\left(\mathrm{d}\sigma^{VS,1}_{\mathrm{NNLO},N_F N}\right)+\mathcal{P}oles\left(\mathrm{d}\sigma^{V,2}_{\mathrm{NNLO},N_F N}\right)=0\,. \tag{9.6}$$

10. Construction of the N_F/N colour factor

The N_F/N colour factor receives contributions from five-parton tree-level $\gamma^*\to q\bar{q}q'\bar{q}'g$, four-parton one-loop $\gamma^*\to q\bar{q}q'\bar{q}'$ and $\gamma^*\to q\bar{q}gg$ at subleading colour as well as three-parton two-loop $\gamma^*\to q\bar{q}g$. The gluon emissions are all photon-like.

This colour factor is part of the QED-type corrections. We described the construction of the subtraction terms for this colour factor previously in [66].

10.1 Five-parton contribution

The NNLO radiation term appropriate for the three jet final state is given by

$$\begin{aligned}\mathrm{d}\sigma^{R}_{\mathrm{NNLO},N_F/N} &= N_5\,\frac{N_F}{N}\,\mathrm{d}\Phi_5(p_1,\ldots,p_5;q)\Big[B_5^{0,c}(1_q,5_g,2_{\bar{q}};3_{q'},4_{\bar{q}'}) \\ &\quad +B_5^{0,d}(1_q,2_{\bar{q}};3_{q'},5_g,4_{\bar{q}'})-2\,B_5^{0,e}(1_q,2_{\bar{q}};3_{q'},4_{\bar{q}'};5_g)\Big]\,J_3^{(5)}(p_1,\ldots,p_5) \\ &= N_5\,\frac{N_F}{N}\,\mathrm{d}\Phi_5(p_1,\ldots,p_5;q)\,\frac{1}{2}\sum_{(i,j)\in(3,4)}\Big[B_5^{0,c}(1_q,5_g,2_{\bar{q}};i_{q'},j_{\bar{q}'}) \\ &\quad +B_5^{0,d}(1_q,2_{\bar{q}};i_{q'},5_g,j_{\bar{q}'})-2\,B_5^{0,e}(1_q,2_{\bar{q}};i_{q'},j_{\bar{q}'};5_g)\Big]\,J_3^{(5)}(p_1,\ldots,p_5)\,,\end{aligned} \tag{10.1}$$

where the symmetrization over the momenta of the secondary quark-antiquark pair exploits the fact that the jet algorithm does not distinguish quarks and antiquarks. This symmetrisation reduces the number of non-vanishing unresolved limits considerably, since the interference term in $B_5^{0,e}$ is odd under this interchange. As a result, the unresolved structure of the symmetrised $B_5^{0,e}$ equals the unresolved structure of $B_5^{0,c}+B_5^{0,d}$.

The subtraction term reads:

$$\begin{aligned}\mathrm{d}\sigma^{S}_{\mathrm{NNLO},N_F/N} &= N_5\,\frac{N_F}{N}\,\mathrm{d}\Phi_5(p_1,\ldots,p_5;q)\,\frac{1}{2}\sum_{(i,j)\in(3,4)}\Bigg\{ \\ &\quad -A_3^0(1_q,5_g,2_{\bar{q}})\,B_4^0(\widetilde{(15)}_q,\widetilde{(25)}_{\bar{q}},i_{q'},j_{\bar{q}'})\,J_3^{(4)}(\widetilde{p_{15}},\widetilde{p_{25}},p_i,p_j) \\ &\quad -A_3^0(i_{q'},5_g,j_{\bar{q}'})\,B_4^0(1_q,2_{\bar{q}},\widetilde{(i5)}_{q'},\widetilde{(j5)}_{\bar{q}'})\,J_3^{(4)}(p_1,p_2,\widetilde{p_{i5}},\widetilde{p_{j5}}) \\ &\quad -\frac{1}{2}\Big\{E_3^0(1_q,i_{q'},j_{\bar{q}'})\,\tilde{A}_4^0(\widetilde{(1j)}_q,\widetilde{(ij)}_g,5_g,2_{\bar{q}})\,J_3^{(4)}(\widetilde{p_{1j}},p_2,\widetilde{p_{ij}},p_5)+(1\leftrightarrow 2)\Big\} \\ &\quad -\bigg(B_4^0(1_q,i_{q'},j_{\bar{q}'},2_{\bar{q}})-\frac{1}{2}\Big\{E_3^0(1_q,i_{q'},j_{\bar{q}'})A_3^0(\widetilde{(1i)}_q,\widetilde{(ji)}_g,2_{\bar{q}})+(1\leftrightarrow 2)\Big\}\bigg) \\ &\quad \times A_3^0(\widetilde{(1ij)}_q,5_g,\widetilde{(2ji)}_{\bar{q}})\,J_3^{(3)}(\widetilde{p_{1ij}},\widetilde{p_{2ji}},p_5) \\ &\quad -\frac{1}{2}\Bigg\{\bigg(\tilde{E}_4^0(1_q,i_{q'},j_{\bar{q}'},5_g)-A_3^0(i_{q'},5_g,j_{\bar{q}'})E_3^0(1_q,\widetilde{(i5)}_{q'},\widetilde{(j5)}_{\bar{q}'})\bigg)\end{aligned}$$

$$\times A_3^0(\widetilde{(1i5)}_q, \widetilde{(j5i)}_g, 2_{\bar{q}})\, J_3^{(3)}(\widetilde{p_{1i5}}, p_2, \widetilde{p_{j5i}}) + (1 \leftrightarrow 2)\Big\}$$

$$+\frac{1}{2}\Big\{E_3^0(1_q, i_{q'}, j_{\bar{q}'})\, A_3^0(\widetilde{(1j)}_q, 5_g, 2_{\bar{q}})\, A_3^0(\widetilde{(\widetilde{(1j)}5)}_q, \widetilde{(ij)}_g, \widetilde{(25)}_{\bar{q}}) \times$$

$$\times J_3^{(3)}(\widetilde{p_{(1j)5}}, \widetilde{p_{25}}, \widetilde{p_{ij}}) + (1 \leftrightarrow 2)\Big\}\Big\} \tag{10.2}$$

10.2 Four-parton contribution

The four parton contribution to the N_F/N colour factor reads:

$$\begin{aligned}
\mathrm{d}\sigma^{V,1}_{\mathrm{NNLO},N_F/N} &= N_4\,\frac{N_F}{N}\left(\frac{\alpha_s}{2\pi}\right)\mathrm{d}\Phi_4(p_1,\ldots,p_4;q)\\
&\quad\Bigg\{-\frac{1}{2}\sum_{(i,j)\in(3,4)}\Big(B_4^{1,b}(1_q, i_{q'}, j_{\bar{q}'}, 2_{\bar{q}}) + 2C_4^{1,f}(1_q, i_q, j_{\bar{q}}, 2_{\bar{q}})\\
&\quad+\tilde{A}_4^{1,c}(1_q, i_g, j_g, 2_{\bar{q}})\Big)\Bigg\}\, J_3^{(4)}(p_1,\ldots,p_4)\,.
\end{aligned} \tag{10.3}$$

Like in the $N_F\,N$ colour factor, the expression is symmetrised over the momenta (3) and (4) to remove terms which are antisymmetric under charge conjugation, and can not be accounted for properly by the quark-gluon antenna functions.

The corresponding subtraction term is:

$$\begin{aligned}
\mathrm{d}\sigma^{VS,1}_{\mathrm{NNLO},N_F/N} &= N_4\,\frac{N_F}{N}\left(\frac{\alpha_s}{2\pi}\right)\mathrm{d}\Phi_4(p_1,\ldots,p_4;q)\\
&\times\Bigg\{\ \Big[\mathcal{A}_3^0(s_{12}) + \mathcal{A}_3^0(s_{34})\Big]\ B_4^0(1_q, 3_{q'}, 4_{\bar{q}'}, 2_{\bar{q}})\, J_3^{(4)}(p_1,\ldots,p_4)\\
&+\frac{1}{4}\Big[\mathcal{E}_3^0(s_{13}) + \mathcal{E}_3^0(s_{14}) + \mathcal{E}_3^0(s_{23}) + \mathcal{E}_3^0(s_{24})\Big]\ \tilde{A}_4^0(1_q, 3_g, 4_g, 2_{\bar{q}})\, J_3^{(4)}(p_1,\ldots,p_4)\\
&-\frac{1}{2}\Bigg\{\Bigg(E_3^0(1_q, 3_{q'}, 4_{\bar{q}'})\Big[\tilde{A}_3^1(\widetilde{(13)}_q, \widetilde{(43)}_g, 2_{\bar{q}}) + \mathcal{A}_2^1(s_{1234})A_3^0(\widetilde{(13)}_q, \widetilde{(43)}_g, 2_{\bar{q}})\Big]\\
&+\mathcal{A}_3^0(s_{\widetilde{(13)}2})\, E_3^0(1_q, 3_{q'}, 4_{\bar{q}'})\, A_3^0(\widetilde{(13)}_q, \widetilde{(43)}_g, 2_{\bar{q}})\\
&+\Big[\tilde{E}_3^1(1_q, 3_{q'}, 4_{\bar{q}'}) + \mathcal{A}_3^0(s_{34})E_3^0(1_q, 3_{q'}, 4_{\bar{q}'})\Big]\ A_3^0(\widetilde{(13)}_q, \widetilde{(43)}_g, 2_{\bar{q}})\Bigg)J_3^{(3)}(\widetilde{p_{13}}, \widetilde{p_{43}}, p_2)\\
&+(1 \leftrightarrow 2)\Bigg\}\\
&-\frac{1}{2}\sum_{(i,j)\in(3,4)}\Bigg(A_3^0(1_q, i_g, 2_{\bar{q}})\hat{A}_3^1(\widetilde{(1i)}_q, j_g, \widetilde{(2i)}_{\bar{q}}) + \Big[\hat{A}_3^1(1_q, i_g, 2_{\bar{q}})\\
&+\frac{1}{2}\left(\mathcal{E}_3^0(s_{1i}) + \mathcal{E}_3^0(s_{1j}) + \mathcal{E}_3^0(s_{2i}) + \mathcal{E}_3^0(s_{2j})\right) A_3^0(1_q, i_g, 2_{\bar{q}})\Big]\ A_3^0(\widetilde{(1i)}_q, j_g, \widetilde{(2i)}_{\bar{q}})\\
&+b_{0,F}\log\frac{q^2}{s_{12i}}A_3^0(1_q, i_g, 2_{\bar{q}})A_3^0(\widetilde{(1i)}_q, j_g, \widetilde{(2i)}_{\bar{q}})\Bigg)J_3^{(3)}(\widetilde{p_{1i}}, \widetilde{p_{2i}}, p_j)\Bigg\}
\end{aligned} \tag{10.4}$$

10.3 Three-parton contribution

The three parton contribution to the N_F/N colour factor consists of the three-parton virtual two-loop correction and the integrated five-parton tree-level and four-parton one-loop subtraction terms, which read

$$\begin{aligned}
&{\rm d}\sigma^{S}_{NNLO,N_F/N} + {\rm d}\sigma^{VS,1}_{NNLO,N_F/N} = \frac{N_F}{N} \\
&\times\Bigg\{ - \Bigg[\mathcal{B}_4^0(s_{12}) + \frac{1}{2}\,\tilde{\mathcal{E}}_4^0(s_{13}) + \frac{1}{2}\,\tilde{\mathcal{E}}_4^0(s_{23}) + \frac{1}{2}\mathcal{A}_3^0(s_{12})\,\left(\mathcal{E}_3^0(s_{13})+\mathcal{E}_3^0(s_{23})\right) \\
&+\hat{\mathcal{A}}_3^1(s_{12}) + \frac{1}{2}\,\tilde{\mathcal{E}}_3^1(s_{13}) + \frac{1}{2}\,\tilde{\mathcal{E}}_3^1(s_{23})\Bigg]\,A_3^0(1_q,3_g,2_{\bar{q}}) - \frac{1}{2}\left(\mathcal{E}_3^0(s_{13})+\mathcal{E}_3^0(s_{23})\right)\,\tilde{A}_3^1(1_q,3_g,2_{\bar{q}}) \\
&-\mathcal{A}_3^0(s_{12})\,\hat{A}_3^1(1_q,3_g,2_{\bar{q}}) - \frac{b_{0,F}}{\epsilon}\,\mathcal{A}_3^0(s_{12})\left((s_{12})^{-\epsilon}-(s_{123})^{-\epsilon}\right)\,A_3^0(1_q,3_g,2_{\bar{q}})\Bigg\}{\rm d}\sigma_3\,. \qquad (10.5)
\end{aligned}$$

Taking the infrared pole part of this expression, we obtain cancellation of all infrared poles in this channel:

$$\mathcal{P}oles\left({\rm d}\sigma^{S}_{NNLO,N_F/N}\right) + \mathcal{P}oles\left({\rm d}\sigma^{VS,1}_{NNLO,N_F/N}\right) + \mathcal{P}oles\left({\rm d}\sigma^{V,2}_{NNLO,N_F/N}\right) = 0\,. \qquad (10.6)$$

11. Construction of the N_F^2 colour factor

The N_F^2 colour factor receives contributions only from the four-parton one-loop process $\gamma^* \to q\bar{q}q'\bar{q}'$ and from the three-parton two-loop process $\gamma^* \to q\bar{q}g$.

This colour factor is also part of the QED-type corrections, described previously in [66].

11.1 Four-parton contribution

The four-parton one-loop contribution to this colour factor is

$${\rm d}\sigma^{V,1}_{NNLO,N_F^2} = N_4\,N_F^2\,\left(\frac{\alpha_s}{2\pi}\right)\,{\rm d}\Phi_4(p_1,\ldots,p_4;q)\,B_4^{1,c}(1_q,3_{q'},4_{\bar{q}'},2_{\bar{q}})\;J_3^{(4)}(p_1,\ldots,p_4). \qquad (11.1)$$

This contribution is free of explicit infrared poles (as can be inferred from the absence of a five-parton contribution to this colour structure).

The subtraction term corresponding to this contribution is

$$\begin{aligned}
{\rm d}\sigma^{VS,1}_{NNLO,N_F^2} &= N_4\,N_F^2\,\left(\frac{\alpha_s}{2\pi}\right)\,{\rm d}\Phi_4(p_1,\ldots,p_4;q)\,\frac{1}{2}\Bigg\{ \\
&\Bigg[\left(\hat{E}_3^1(1_q,3_{q'},4_{\bar{q}'}) + b_{0,F}\,\log\frac{q^2}{s_{134}}\,E_3^0(1_q,3_{q'},4_{\bar{q}'})\right)\,A_3^0(\widetilde{(13)}_q,\widetilde{(43)}_g,2_{\bar{q}}) \\
&+E_3^0(1_q,3_{q'},4_{\bar{q}'})\,\hat{A}_3^1(\widetilde{(13)}_q,\widetilde{(43)}_g,2_{\bar{q}})\Bigg]\,J_3^{(3)}(\widetilde{p_{13}},\widetilde{p_{43}},p_2) \\
&+\Bigg[\left(\hat{E}_3^1(2_{\bar{q}},3_{q'},4_{\bar{q}'}) + b_{0,F}\,\log\frac{q^2}{s_{234}}\,E_3^0(2_{\bar{q}},3_{q'},4_{\bar{q}'})\right)\,A_3^0(1_q,\widetilde{(43)}_g,\widetilde{(23)}_{\bar{q}}) \\
&+E_3^0(2_{\bar{q}},3_{q'},4_{\bar{q}'})\,\hat{A}_3^1(1_q,\widetilde{(43)}_g,\widetilde{(23)}_{\bar{q}})\Bigg]\,J_3^{(3)}(p_1,\widetilde{p_{43}},\widetilde{p_{23}})\Bigg\}\,. \qquad (11.2)
\end{aligned}$$

Although $\hat{E}_3^1$ and $\hat{A}_3^1$ contain explicit infrared poles, these cancel in their sum, as can be seen from (5.16) and (6.32) of [32]. $\mathrm{d}\sigma^{VS,1}_{\mathrm{NNLO},N_F^2}$ is therefore free of explicit infrared poles.

11.2 Three-parton contribution

The three parton contribution to the N_F^2 colour factor consists of the three-parton virtual two-loop correction and the integrated four-parton one-loop subtraction term, which reads

$$\begin{aligned}
\mathrm{d}\sigma^{VS,1}_{\mathrm{NNLO},N_F^2} = N_F^2 \, \frac{1}{2} \Bigg[& \left(\hat{\mathcal{E}}_3^1(s_{13}) + \hat{\mathcal{E}}_3^1(s_{23})\right) \; A_3^0(1_q, 3_g, 2_{\bar{q}}) + \\
& \left(\mathcal{E}_3^0(s_{13}) + \mathcal{E}_3^0(s_{23})\right) \; \hat{A}_3^1(1_q, 3_g, 2_{\bar{q}}) + \frac{b_{0,F}}{\epsilon} \Big[\mathcal{E}_3^0(s_{13}) \left((s_{13})^{-\epsilon} - (s_{123})^{-\epsilon}\right) \\
& + \mathcal{E}_3^0(s_{23}) \left((s_{23})^{-\epsilon} - (s_{123})^{-\epsilon}\right) \Big] A_3^0(1_q, 3_g, 2_{\bar{q}}) \Bigg] \mathrm{d}\sigma_3 \, . \qquad (11.3)
\end{aligned}$$

Combining the infrared poles of this expression with the two loop matrix element, we obtain the cancellation of all infrared poles in this colour factor,

$$\mathcal{P}oles\left(\mathrm{d}\sigma^{VS,1}_{\mathrm{NNLO},N_F^2}\right) + \mathcal{P}oles\left(\mathrm{d}\sigma^{V,2}_{\mathrm{NNLO},N_F^2}\right) = 0 \, . \qquad (11.4)$$

12. Construction of the $N_{F,\gamma}$ colour factor

The $N_{F,\gamma}$ colour factor comes from the interference of amplitudes in which the external vector boson couples to different quark lines. It receives contributions from five-parton tree-level $\gamma^* \to q\bar{q}q'\bar{q}'g$, four-parton one-loop $\gamma^* \to q\bar{q}q'\bar{q}'$ and $\gamma^* \to q\bar{q}gg$ as well as three-parton two-loop $\gamma^* \to q\bar{q}g$ and $\gamma^* \to ggg$. This colour factor is absent in three-jet production at NLO and four-jet production at LO because of Furry's theorem [5], and its contribution to four-jet production at NLO is numerically tiny [52]. The numerical magnitude of this term in the two-loop corrections to the three-parton channel is equally very small [18, 53]. A detailed discussion of this colour factor, and of the effects leading to its numerical suppression is contained in [52]. All partonic channels contributing to this colour factor are individually finite.

In this section, we document this colour factor for completeness. Given that is numerical impact can be safely expected to be negligible, we refrain from its implementation.

12.1 Five-parton contribution

The NNLO radiation term appropriate for the three jet final state is given by

$$\begin{aligned}
\mathrm{d}\sigma^{R}_{\mathrm{NNLO},N_{F,\gamma}} = N_5 \, N_{F,\gamma} \, \mathrm{d}\Phi_5(p_1, \ldots, p_5; q) \Bigg[& - N \Big(\hat{B}_5^{0,a}(1_q, 5_g, 4_{\bar{q}'}; 3_{q'}, 2_{\bar{q}}) \\
& + \hat{B}_5^{0,b}(1_q, 4_{\bar{q}'}; 3_{q'}, 5_g, 2_{\bar{q}}) - \hat{B}_5^{0,e}(1_q, 4_{\bar{q}'}; 3_{q'}, 2_{\bar{q}}, 5_g) \Big) \\
& + \frac{1}{N} \Big(\hat{B}_5^{0,c}(1_q, 5_g, 2_{\bar{q}}; 3_{q'}, 4_{\bar{q}'}) + \hat{B}_5^{0,d}(1_q, 2_{\bar{q}}; 3_{q'}, 5_g, 4_{\bar{q}'}) \\
& + \hat{B}_5^{0,e}(1_q, 2_{\bar{q}}; 3_{q'}, 4_{\bar{q}'}; 5_g) \Big) \Bigg] J_3^{(5)}(p_1, \ldots, p_5) \, . \qquad (12.1)
\end{aligned}$$

Once symmetrised over the quark and antiquark momenta, this term can be integrated safely without the need for an infrared subtraction. It is free from infrared singularities associated with gluon 5_g unresolved, since the corresponding four-parton tree-level term $\hat{B}_4^0$ vanishes after symmetrisation over the quark and antiquark momenta. Double unresolved singularities can not appear since there is no tree-level three-parton process proportional to $N_{F,\gamma}$.

12.2 Four-parton contribution

The four parton contribution to the $N_{F,\gamma}$ colour factor reads:

$$\begin{aligned}
\mathrm{d}\sigma^{V,1}_{\mathrm{NNLO},N_{F,\gamma}} &= N_4\, N_{F,\gamma} \left(\frac{\alpha_s}{2\pi}\right) \mathrm{d}\Phi_4(p_1,\ldots,p_4;q) \\
&\quad\times \Bigg\{ N\hat{B}_4^{1,a}(1_q,3_q,4_{\bar{q}},2_{\bar{q}}) - \frac{1}{N}\hat{B}_4^{1,b}(1_q,3_q,4_{\bar{q}},2_{\bar{q}}) \\
&\quad +N\, A_4^{1,e}(1_q,i_g,j_g,2_{\bar{q}}) - \frac{1}{N}\tilde{A}_4^{1,e}(1_q,3_g,4_g,2_{\bar{q}}) \Bigg\}\, J_3^{(4)}(p_1,\ldots,p_4)\,.
\end{aligned} \tag{12.2}$$

Terms which vanish under symmetrisation of the quark and antiquark momenta, arising from $\hat{B}_4^{1,c}$ in (4.46) have been omitted here. After this symmetrisation, all explicit infrared poles present in individual terms in the above expression cancel. Moreover, (12.2) is finite in all single unresolved limits, such that no antenna subtraction is needed.

12.3 Three-parton contribution

The three-parton contribution to the $N_{F,\gamma}$ colour factor consists of the three-parton virtual two-loop correction to $\gamma^* \to q\bar{q}g$ [18] and the one-loop squared correction to $\gamma^* \to ggg$ [53]. Both are individually finite, and were shown to be numerically tiny. Since no subtractions were carried out in the five-parton and four-parton channels, there are no integrated subtraction terms in the three-parton channel.

13. Numerical implementation

Using the matrix elements and antenna subtraction terms derived in the previous sections, NNLO corrections to any infrared-safe three-jet observable in e^+e^- annihilation (jet cross section, event shape variable) can be computed numerically. We implemented this numerical evaluation into a parton-level event generator program, which we name `EERAD3`.

This program is based on the program `EERAD2` [39], which computes four-jet production at NLO. `EERAD2` contained already the five-parton and four-parton matrix elements relevant here, as well as the NLO-type subtraction terms $\mathrm{d}\sigma^{S,a}_{\mathrm{NNLO}}$ and $\mathrm{d}\sigma^{VS,1,a}_{\mathrm{NNLO}}$.

The implementation contains three channels, classified by their partonic multiplicity:

- in the five-parton channel, we integrate

$$\mathrm{d}\sigma^{R}_{\mathrm{NNLO}} - \mathrm{d}\sigma^{S}_{\mathrm{NNLO}}\,. \tag{13.1}$$

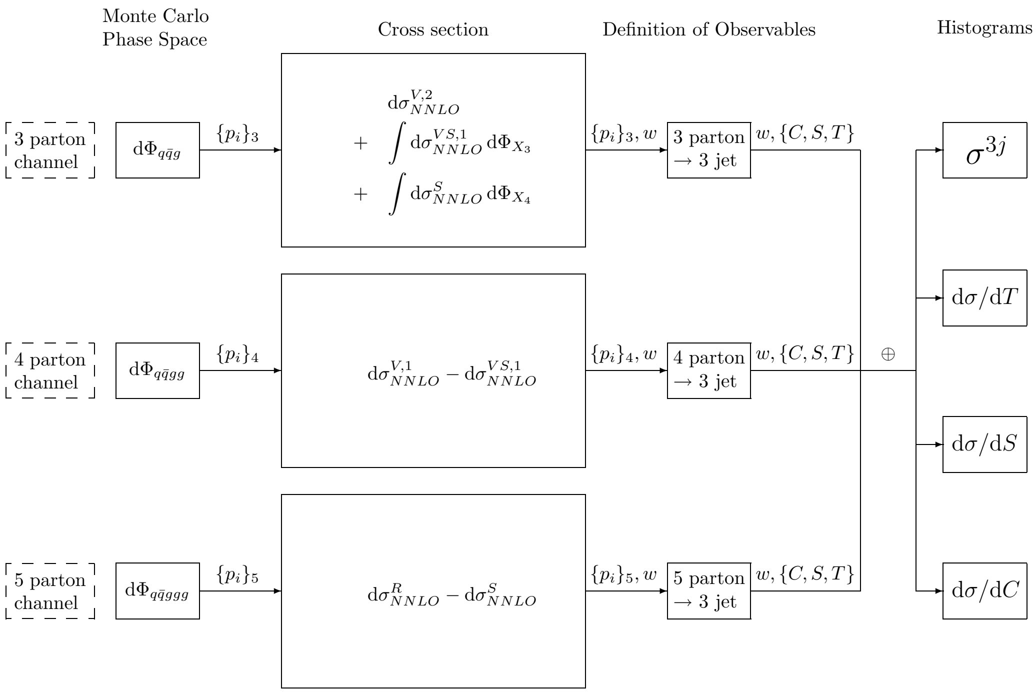

Figure 7: Structure of the EERAD3 parton-level Monte Carlo event generator programme.

- in the four-parton channel, we integrate

$$\mathrm{d}\sigma^{V,1}_{\mathrm{NNLO}} - \mathrm{d}\sigma^{VS,1}_{\mathrm{NNLO}}\,. \tag{13.2}$$

- in the three-parton channel, we integrate

$$\mathrm{d}\sigma^{V,2}_{\mathrm{NNLO}} + \mathrm{d}\sigma^{S}_{\mathrm{NNLO}} + \mathrm{d}\sigma^{VS,1}_{\mathrm{NNLO}}\,. \tag{13.3}$$

The numerical integration over these channels is carried out by Monte Carlo methods using the `VEGAS` [67] implementation. The structure of the programme is displayed in figure 7.

The phase space in the four-parton and five-parton channel is decomposed into wedges which are constructed such that two of the invariants are smaller than any of the other invariants. This decomposition allows an optimal generation of phase space points in the unresolved limits. The full four-parton phase space is obtained by summing (a) 12 wedges with (s_{ij}, s_{ik}) smallest, plus (b) 3 wedges with (s_{ij}, s_{kl}) smallest. To obtain the full five-parton phase space, we sum (a) 30 wedges with (s_{ij}, s_{ik}) smallest, and (b) 15 wedges with (s_{ij}, s_{kl}) smallest. The phase space integration in either channel is carried out by integrating only over a single wedge of type (a) and a single wedge of type (b), while summing the integrands appropriate to all wedges of the given type. In doing this summation, we combine (in the exact unresolved limits) phase space points which are related to each other by a rotation of the system of unresolved partons, thereby largely cancelling the angular-dependent terms. In all colour factors containing angular-dependent terms, the combination of phase space wedges yields a substantial improvement of the numerical stability of the results.

It was already demonstrated above that the integrands in the four-parton and three-parton channel are free of explicit infrared poles. In the five-parton and four-parton channel, we tested the proper implementation of the subtraction by generating trajectories of phase space points approaching a given single or double unresolved limit using the RAMBO [68] phase space generator. Along these trajectories, we observe that the antenna subtraction terms converge towards the physical matrix elements, and that the cancellations among individual contributions to the subtraction terms take place as expected in the antenna subtraction method.

Moreover, we checked the correctness of the subtraction by introducing a lower cut (slicing parameter) y_0 on all phase space variables, and observing that our results are independent of this cut (provided it is chosen small enough). This behaviour indicates that the subtraction terms ensure that the contribution of potentially singular regions of the final state phase space does not contribute to the numerical integrals, but is accounted for analytically.

14. Thrust distribution as an example

To illustrate the implementation and to study the numerical impact of the individual NNLO contributions, we consider the thrust distribution. We already reported the NNLO results on this event shape distribution in a previous paper [33], where the phenomenological implications are discussed in detail.

The thrust variable for a hadronic final state in e^+e^- annihilation is defined as [69]

$$T = \max_{\vec{n}} \left(\frac{\sum_i |\vec{p_i} \cdot \vec{n}|}{\sum_i |\vec{p_i}|} \right) , \tag{14.1}$$

where p_i denotes the three-momentum of particle i, with the sum running over all particles. The unit vector $\vec{n}$ is varied to find the thrust direction $\vec{n}_T$ which maximises the expression in parentheses on the right hand side.

It can be seen that a two-particle final state has fixed $T = 1$, consequently the thrust distribution receives its first non-trivial contribution from three-particle final states, which, at order α_s, correspond to three-parton final states. Therefore, both theoretically and experimentally, the thrust distribution is closely related to three-jet production.

A study of the phenomenological implications of the NNLO corrections to the thrust distribution was presented in [33], illustrating that the NNLO corrections amount to about 15% of the total result over the experimentally relevant range $0.02 < 1-T < 0.3$, and that inclusion of these corrections results in a considerable stabilization of the renormalisation scale dependence of the theoretical prediction. In the present context, we use the thrust distribution only as an example to illustrate certain features of our calculation.

The three-jet rate and event shapes related to it can be expressed in perturbative QCD by dimensionless coefficients. These coefficients depend, for non-singlet QCD corrections, only on the jet resolution parameter (respectively on the event shape variable). Typically, one denotes these coefficients by $A, B, C, \ldots$ at LO, NLO, NNLO, etc.

The perturbative expansion of thrust distribution up to NNLO for renormalisation scale $\mu^2 = s$ and $\alpha_s \equiv \alpha_s(s)$ is then given by

$$\frac{1}{\sigma^{\rm had}}\frac{{\rm d}\sigma}{{\rm d}T} = \left(\frac{\alpha_s}{2\pi}\right)\frac{{\rm d}\bar{A}}{{\rm d}T} + \left(\frac{\alpha_s}{2\pi}\right)^2\frac{{\rm d}\bar{B}}{{\rm d}T} + \left(\frac{\alpha_s}{2\pi}\right)^3\frac{{\rm d}\bar{C}}{{\rm d}T}\,. \tag{14.2}$$

Here we define the effective coefficients in terms of the perturbatively calculated coefficients A, B and C, which are all normalised to the tree-level cross section

$$\sigma_0 = \frac{4\pi\alpha}{3s} N\, e_q^2\,. \tag{14.3}$$

for $e^+e^- \to q\bar{q}$. Using

$$\sigma_{\rm had} = \sigma_0\left(1 + \frac{3}{2}C_F\left(\frac{\alpha_s}{2\pi}\right) + K_2\left(\frac{\alpha_s}{2\pi}\right)^2 + \mathcal{O}(\alpha_s^3)\right)\,, \tag{14.4}$$

with ($C_F = (N^2-1)/(2N)$, $C_A = N$, $T_R = 1/2$ for $N = 3$ colours and N_F light quark flavours)

$$K_2 = \frac{1}{4}\left[-\frac{3}{2}C_F^2 + C_F C_A\left(\frac{123}{2} - 44\zeta_3\right) + C_F T_R N_F\left(-22 + 16\zeta_3\right)\right]\,, \tag{14.5}$$

we obtain:

$$\begin{aligned}\bar{A} &= A\,,\\ \bar{B} &= B - \frac{3}{2}C_F\,A\,,\\ \bar{C} &= C - \frac{3}{2}C_F\,B + \left(\frac{9}{4}C_F^2 - K_2\right)\,A\,.\end{aligned} \tag{14.6}$$

These coefficients depend only on the jet resolution parameter or the event shape variable under consideration, and are independent of electroweak couplings, centre-of-mass energy and renormalisation scale.

The above coefficients include only QCD corrections with non-singlet quark couplings. At $\mathcal{O}(\alpha_s^2)$, these amount to the full corrections, while the $\mathcal{O}(\alpha_s^3)$ corrections also receive a singlet contribution. As discussed above, this singlet contribution arises from the interference of diagrams where the external gauge boson couples to different quark lines. In four-jet observables at $\mathcal{O}(\alpha_s^3)$, these singlet contributions were found to be extremely small [52]. Also, the singlet contribution from three-gluon final states to three-jet observables was found to be negligible [53].

We determine A, B, C from the perturbative contributions to the differential cross section, normalised to the tree-level hadronic cross section:

$$\frac{{\rm d}A}{{\rm d}T} = \frac{1}{\sigma_0}\frac{{\rm d}\sigma_{\rm LO}}{{\rm d}T}\,, \qquad \frac{{\rm d}B}{{\rm d}T} = \frac{1}{\sigma_0}\frac{{\rm d}\sigma_{\rm NLO}}{{\rm d}T}\,, \qquad \frac{{\rm d}C}{{\rm d}T} = \frac{1}{\sigma_0}\frac{{\rm d}\sigma_{\rm NNLO}}{{\rm d}T}\,. \tag{14.7}$$

For the determination of the non-singlet coefficients, it is sufficient to consider σ_0 for pure photon exchange, since any electroweak coupling constant cancels out in the above ratio.

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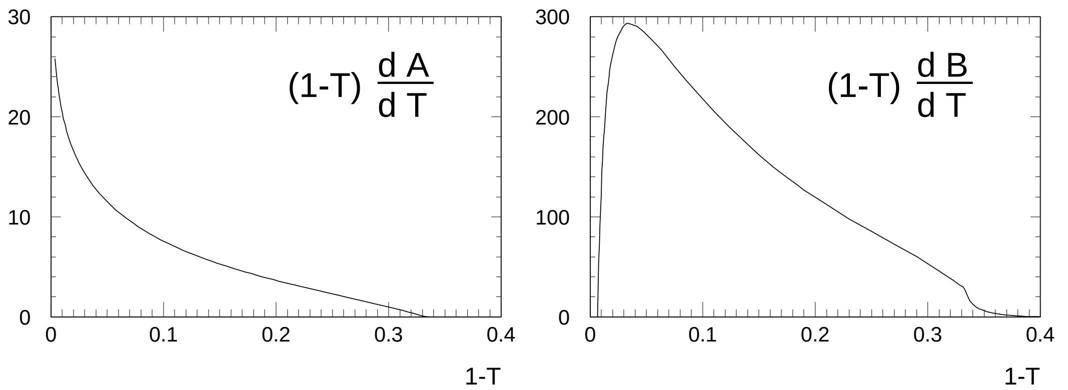

Figure 8: Coefficients of the leading order and next-to-leading order contributions to the thrust distributions.

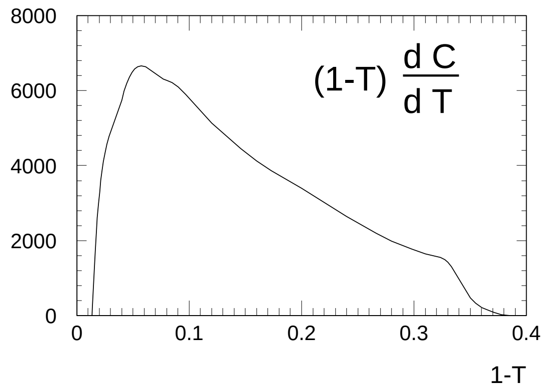

Figure 9: Coefficient of the next-to-next-to-leading order contribution to the thrust distribution.

The LO and NLO coefficients $A(T)$ and $B(T)$ were computed in the literature long ago [5–8, 10]. They are displayed for comparison in figure 8.

The total NNLO coefficient $C(T)$ is displayed in figure 9. The six different colour factor contributions to it are shown in figure 10. It can be seen that the numerically dominant contributions come from the N^2 and $N_F N$ colour factors. The contributions of these two colour factors are of opposite sign, with N^2 being of larger absolute magnitude, thus resulting in a total positive result. Contributions at the 10% level of the total come from the N_F^2 and N^0 colour factors, N_F/N amounts to about 5%, while the most subleading $1/N^2$ colour factor is below 1%.

To illustrate the independence of our results on y_0, we display the different colour factor contributions to $C(T)$ as function of $-\ln(1-T)$ in figure 11 for different values of the phase space cut $y_0 = 10^{-5}, 10^{-6}, 10^{-7}$. By rescaling all phase space invariants to the total centre-of-mass energy squared, y_0 becomes dimensionless. Since the value of $(1-T)$ determines the typical scale of the smallest resolved invariant, one must require y_0 to be several orders of magnitude smaller than $(1-T)$ for the cancellation between matrix element and subtraction term to be accurate. Figure 11 shows very clearly that over the phenomenologically relevant

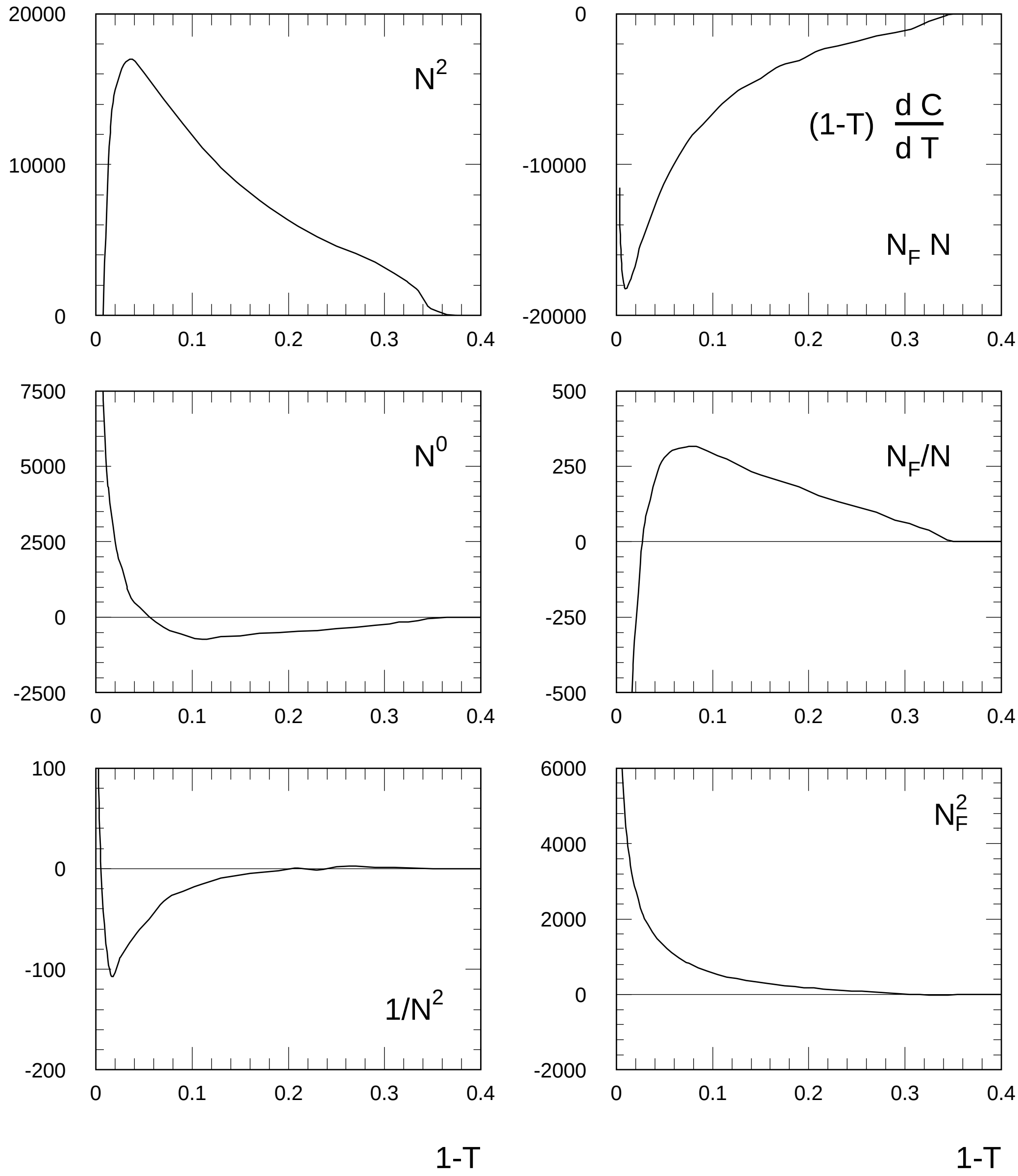

Figure 10: Different colour factor contributions to NNLO coefficient of the thrust distribution.

range, i.e. $0.02 < 1 - T$ or equivalently, $-\ln(1 - T) < 4$, our results do not depend on y_0. As $(1-T)$ approaches y_0 (starting at about $(1-T) \approx \mathcal{O}(1000) y_0$), the calculation becomes unreliable as expected. This behaviour can be understood to arise from the fact that the subtraction terms converge to the full matrix element only once all unresolved invariants are much smaller than any of the resolved invariants.

The numerical convergence of our calculation deteriorates for lower values of y_0 for two reasons.

- the absolute magnitude of matrix elements and subtraction terms increases for decreasing y_0 both in the five-parton and four-parton channel. Consequently, numerical

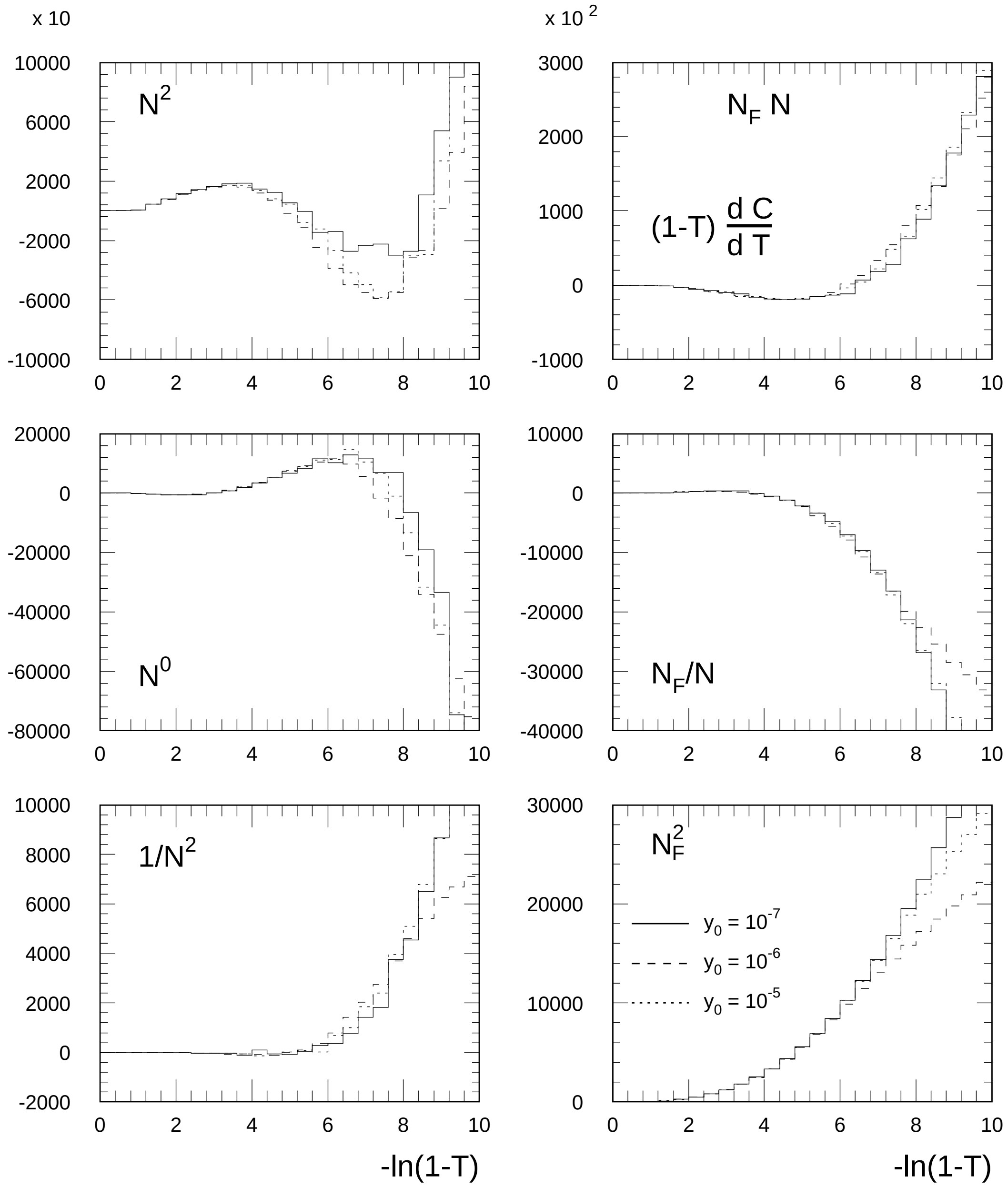

Figure 11: Dependence on phase space cut y_0 in different colour factors. We see that the results are independent of y_0 for $-\ln(1-T) < 4$, but, as explained in the text, differ at larger values of $-\ln(1-T)$

cancellations between matrix elements and subtraction terms happen over larger orders of magnitude, thereby enhancing numerical rounding errors.

- the four-parton one-loop matrix elements start themselves to become numerically unstable because of the presence of inverse Gram determinants, which can become singular inside the integration region.

Therefore, for all phenomenological applications to the thrust distribution [33], we choose $y_0 = 10^{-5}$. For applications to other event shapes, one expects a similar behaviour, and one must first determine the value of y_0 required for reliable predictions in the phenomenologically relevant range for that observable.

15. Conclusions and outlook

In this paper, we provide a detailed description of the calculation of NNLO QCD corrections to three-jet production and related event shapes in electron-positron annihilation. At this order, three-parton, four-parton and five-parton subprocesses contribute. The three-parton and four-parton subprocesses contain explicit infrared singularities from loop corrections. Four-parton and five-parton subprocesses contain singularities which only become explicit after integrating the contributions over the phase space relevant to the three-jet final states. Those singularities arise when one or two partons become unresolved (collinear or soft). For an infrared-safe observable, adding together all infrared singularities, one observes a complete cancellation, resulting in an infrared-finite result.

To implement the four-parton and five-parton processes in a numerical programme, one has to devise a procedure for extracting the implicit infrared singularities from them. We extract these singularities using subtraction terms which numerically subtract all infrared singularities from the five-parton and four-parton channels. The subtraction terms are then integrated analytically and combined with the three-parton channel, where they cancel all explicit infrared poles. The subtraction terms are derived using the antenna subtraction method, which is based on antenna functions encapsulating all unresolved partonic radiation emitted from a pair of hard radiator partons.

Three-jet production at NNLO receives contributions from seven different colour factors. Among these, only the six colour factors of non-singlet configurations require subtraction, while the singlet colour factor is separately finite in all three partonic channels. We describe the construction of the antenna subtraction terms for the six non-singlet colour factors in detail, and demonstrate the cancellation of infrared poles.

All partonic channels have been implemented in a parton-level event generator programme `EERAD3`, which can be used to compute any infrared-safe observable related to three-jet final states in e^+e^- annihilation. We devised various tests of the implementation, demonstrating in particular the correct numerical cancellation between matrix elements and subtraction terms and the independence on phase space restrictions deep inside the subtraction regions.

We observe that the largest part of the NNLO correction is contained in the two leading colour factors N^2 and $N_F N$. The remaining four colour factors yield corrections at or below the ten per cent level.

First phenomenological results on the thrust distribution at NNLO were obtained already in an earlier paper [33]. In the thrust distribution, the NNLO corrections amount to about 15% of the total result. They are lower in magnitude than the NLO corrections, indicating the perturbative stability of this observable. Inclusion of the NNLO corrections considerably reduces the dependence of the result on the renormalisation scale.

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At LEP, a wide variety of QCD event shapes was measured to high precision [70]. The accurate extraction of the strong coupling constant α_s from these data sets was up to now limited by the theoretical uncertainty inherent to the available NLO calculations. We expect that our new NNLO results will improve this situation considerably. Numerical studies of other event shape variables and of the three-jet rate are ongoing, and will be reported elsewhere.

On approaching the two-jet limit ($(1-T) \to 0$ in the thrust distribution), one observes that the perturbative fixed-order expansion starts to break down due to the emergence of large logarithmic corrections at all orders in perturbation theory. By matching NLO calculations and NLL resummation of these logarithmic corrections [11], a reliable description of distributions over the full kinematic range could be accomplished. Such a matching is also possible at NNLO, where it requires the derivation of a number of new matching constants for each event shape.

The subtraction terms derived here for $e^+e^- \to 3$ jets at NNLO can be transcribed to the crossed reactions $ep \to (2+1)$ jets and $pp \to V + 1$ jet at NNLO without much modification. In these cases, which involve partons in the initial state [41], the same antenna functions are used with different antenna phase spaces. To accomplish the above-mentioned NNLO calculations therefore still requires the analytical integration of the relevant antenna functions over the phase spaces relevant to their initial-state kinematics, which appears feasible with present technology. NNLO calculations of other exclusive observables at hadron colliders, such as $pp \to 2$ jets, could also be carried out using the antenna subtraction method by constructing their subtraction terms along the lines described here.

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Erratum

Our results on NNLO corrections to three-jet-like observables [71] were checked by two subsequent studies: the calculation of all logarithmically enhanced contributions to the thrust distribution by Becher and Schwartz [72], and an independent implementation of our subtraction formulae by Weinzierl [73].

These works uncovered numerical discrepancies in the two-jet limit of the observables in two of the six colour factor contributions: N^2 and N^0. In [73], it was shown that the origin of these discrepancies is in an oversubtraction of large-angle soft radiation. In this erratum, we explain the corrected treatment of large-angle soft radiation and its numerical implementation. We discuss the numerical impact of the correction terms.

The modifications concern only sections 3 (treatment of angular terms) and 14 (numerical results).

3. Phase space mappings

3.3 Decomposition of antenna functions into sub-antennae

There were typographical errors in the decomposition of the D_4^0 antenna function into sub-antennae, the correct decomposition reads

$$\begin{aligned}
D_{4,a}^0(1,3,4,5) &= \frac{1}{2}D_{4,l}^0(1,3,4,5) + A_4^0(1,3,4,5) - \frac{1}{2}\tilde{E}_4^0(1,3,5,4) \\
&\quad + A_3^0(1,3,4)\, S_3^0(\widetilde{(13)},\widetilde{(43)},5) \\
&\quad + \frac{1}{2}A_3^0(3,4,5)\,\Big(S_3^0(1,\widetilde{(54)},\widetilde{(34)}) - R_3^0(1,\widetilde{(54)},\widetilde{(34)})\Big) \\
&\quad - E_3^0(5,4,3)Q_3^0(1,\widetilde{(34)},\widetilde{(54)}) - A_3^0(1,3,4)\, E_3^0(\widetilde{(13)},\widetilde{(43)},5) \\
&\quad - A_3^0(1,3,4)\, Q_3^0(\widetilde{(13)},5,\widetilde{(43)})\ , \\
D_{4,b}^0(1,3,4,5) &= D_{4,a}^0(1,5,4,3)\ , \\
D_{4,c}^0(1,3,4,5) &= \tilde{A}_{4,a}^0(1,3,5,4) - E_4^0(1,5,4,3) + B_4^0(1,5,4,3) + C_4^0(1,4,5,3) \\
&\quad + E_3^0(3,4,5)Q_3^0(1,\widetilde{(54)},\widetilde{(34)}) + A_3^0(1,3,4)\, E_3^0(\widetilde{(13)},\widetilde{(43)},5) \\
&\quad + a_3^0(1,3,4)\, Q_3^0(\widetilde{(13)},5,\widetilde{(43)}) + a_3^0(4,5,1)\, Q_3^0(\widetilde{(15)},3,\widetilde{(45)})\ , \\
D_{4,d}^0(1,3,4,5) &= D_{4,c}^0(1,5,4,3)\ .
\end{aligned} \tag{3.23}$$

Our numerical implementation contained already the correct formulae.

3.4 Angular terms

The angular terms in the single unresolved limits are associated with a gluon splitting into two gluons or into a quark-antiquark pair. They average to zero after integration over the antenna phase space. To ensure numerical stability and reliability, this average has to take place within each phase space mapping. We have checked this to be the case for the decompositions of E_4^0 and D_4^0 in (3.18), (3.23) of [71]. The angular average in single collinear limits can be made using the standard momentum parametrisation [10,51] for the $i \parallel j$ limit:

$$p_i^\mu = zp^\mu + k_\perp^\mu - \frac{k_\perp^2}{z}\frac{n^\mu}{2p\cdot n}\,, \qquad p_j^\mu = (1-z)p^\mu - k_\perp^\mu - \frac{k_\perp^2}{1-z}\frac{n^\mu}{2p\cdot n}\,,$$
$$\text{with } 2p_i\cdot p_j = -\frac{k_\perp^2}{z(1-z)}\,, \qquad p^2 = n^2 = 0\,.$$

In this p^μ denotes the collinear momentum direction, and n^μ is an auxiliary vector. The collinear limit is approached by $k_\perp^2 \to 0$.

In the simple collinear $i \parallel j$ limit of the four-parton antenna functions $X_4^0(i,j,k,l)$, one chooses $n = p_k$ to be one of the non-collinear momenta, such that the antenna function can be expressed in terms of p, n, $k_\perp$ and p_l. Expanding in $k_\perp^\mu$ yields only non-vanishing scalar products of the form $p_l \cdot k_\perp$. Expressing the integral over the antenna phase space in the (p,n) centre-of-mass frame, the angular average can be carried out as

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi\,(p_l\cdot k_\perp) = 0\,, \qquad \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi\,(p_l\cdot k_\perp)^2 = -k_\perp^2\,\frac{p\cdot p_l\, n\cdot p_l}{p\cdot n}\,. \tag{3.27}$$

Higher powers of $k_\perp^\mu$ are not sufficiently singular to contribute to the collinear limit. Using the above average, we could analytically verify the cancellation of angular terms within each single phase space mapping, which is independent on the choice of the reference vector n_μ.

In the N^2 and N^0 colour factor, the angular averaging is not sufficient to cancel the $1/\epsilon$ poles in the four-parton one loop subtraction terms [73]. As mentioned in [71] in either of these colour factors, the difference $\mathrm{d}\sigma_{NNLO}^{V,1} - \mathrm{d}\sigma_{NNLO}^{VS,1}$ contains left-over poles of the form

$$\frac{1}{\epsilon^2}X_3^0(1,i,2)\,Y_3^0(\widetilde{1i},j,\widetilde{2i})J_3^{(3)}(\widetilde{1i},j,\widetilde{2i})\left\{s_{\widetilde{1i}j}^{-\epsilon} + s_{\widetilde{2i}j}^{-\epsilon} - s_{1j}^{-\epsilon} - s_{2j}^{-\epsilon} - s_{1i2}^{-\epsilon} + s_{12}^{-\epsilon}\right\}\,, \tag{3.28}$$

where X_3^0 and Y_3^0 are tree-level three-parton antenna functions. Contrary to statements made in [73], these terms do not appear in the colour factor $N_F\,N$ in our implementation.

Furthermore, for these two colour factors the five-parton subtraction terms themselves do introduce spurious limits from large angle soft radiation. The single soft limit of i or k in (6.3) in [71] is non-vanishing. Instead, it yields (soft i):

$$\begin{aligned}&+\frac{1}{2}d_3^0(2,k,j)A_3^0(1,\widetilde{jk},\widetilde{2k})\left[S_{1i(\widetilde{jk})} + S_{1i(\widetilde{2k})} - S_{(\widetilde{2k})i(\widetilde{jk})} - S_{1ij} - S_{1i2} + S_{2ij}\right] + (1\leftrightarrow 2)\\ &-\frac{1}{2}A_3^0(1,k,2)A_3^0((\widetilde{1k}),j,(\widetilde{2k}))\left[S_{(\widetilde{1k})ij} + S_{(\widetilde{2k})ij} - S_{(\widetilde{1k})i(\widetilde{2k})} - S_{1ij} - S_{2ij} + S_{1i2}\right]\end{aligned} \tag{3.29}$$

with

$$S_{abc} = 2\frac{s_{ac}}{s_{ab}s_{bc}} \tag{3.30}$$

To account for this large angle soft radiation, a new subtraction term $\mathrm{d}\sigma_{NNLO}^{A}$ is introduced. This term is added to the five-parton subtraction term $\mathrm{d}\sigma_{NNLO}^{S}$, and its integrated form is subtracted from the four-parton subtraction term $\mathrm{d}\sigma_{NNLO}^{VS,1}$, cancelling the left - over $1/\epsilon$ terms and adding new finite contributions to the four-parton and the five-parton subtraction term.

The new subtraction term $\mathrm{d}\sigma^A_{NNLO}$ contributes only in the N^2 and N^0 colour factors. Its contribution to N^2 reads:

$$
\begin{aligned}
\mathrm{d}\sigma^A_{NNLO,N^2} = {} & N_5\, N^2\, \mathrm{d}\Phi_5(p_1,\ldots,p_5;q)\, \frac{1}{3!} \sum_{(i,j,k)\in P_C(3,4,5)} \Bigg\{ \\
& +\frac{1}{2}\Big(S_{\widetilde{((1i)k)}i\widetilde{((ji)k)}} - S_{\widetilde{(1i)}i\widetilde{(ji)}} - S_{2i\widetilde{((ji)k)}} + S_{2i\widetilde{(ji)}} - S_{2i\widetilde{((1i)k)}} + S_{2i\widetilde{(1i)}} \Big) \\
& \quad \times d_3^0(\widetilde{(1i)}_q, k_g, \widetilde{(ji)}_g)\, A_3^0(\widetilde{((1i)k)}_q, \widetilde{((ji)k)}_g, 2_{\bar q})\, J_3^{(3)}(\widetilde{p_{(1i)k}}, \widetilde{p_{(ji)k}}, p_2) \\
& +\frac{1}{2}\Big(S_{\widetilde{((1k)i)}k\widetilde{((jk)i)}} - S_{\widetilde{(1k)}k\widetilde{(jk)}} - S_{2k\widetilde{((jk)i)}} + S_{2k\widetilde{(jk)}} - S_{2k\widetilde{((1k)i)}} + S_{2k\widetilde{(1k)}} \Big) \\
& \quad \times d_3^0(\widetilde{(1k)}_{\bar q}, i_g, \widetilde{(jk)}_g)\, A_3^0(\widetilde{((1k)i)}_q, \widetilde{((jk)i)}_g, 2_{\bar q})\, J_3^{(3)}(\widetilde{p_{(1k)i}}, \widetilde{p_{(jk)i}}, p_2) \\
& +\frac{1}{2}\Big(S_{\widetilde{((2i)k)}i\widetilde{((ji)k)}} - S_{\widetilde{(2i)}i\widetilde{(ji)}} - S_{1i\widetilde{((ji)k)}} + S_{1i\widetilde{(ji)}} - S_{1i\widetilde{((2i)k)}} + S_{1i\widetilde{(2i)}} \Big) \\
& \quad \times d_3^0(\widetilde{(2i)}_{\bar q}, k_g, \widetilde{(ji)}_g)\, A_3^0(1_q, \widetilde{((ji)k)}_g, \widetilde{((2i)k)}_{\bar q})\, J_3^{(3)}(p_1, \widetilde{p_{(ji)k}}, \widetilde{p_{(2i)k}}) \\
& +\frac{1}{2}\Big(S_{\widetilde{((2k)i)}k\widetilde{((jk)i)}} - S_{\widetilde{(2k)}k\widetilde{(jk)}} - S_{1k\widetilde{((jk)i)}} + S_{1k\widetilde{(jk)}} - S_{1k\widetilde{((2k)i)}} + S_{1k\widetilde{(2k)}} \Big) \\
& \quad \times d_3^0(\widetilde{(2k)}_{\bar q}, i_g, \widetilde{(jk)}_g)\, A_3^0(1_q, \widetilde{((jk)i)}_g, \widetilde{((2k)i)}_{\bar q})\, J_3^{(3)}(p_1, \widetilde{p_{(jk)i}}, \widetilde{p_{(2k)i}}) \\
& -\frac{1}{2}\Big(S_{\widetilde{((1i)k)}i\widetilde{((2i)k)}} - S_{\widetilde{((1i)k)}ij} - S_{\widetilde{((2i)k)}ij} + S_{\widetilde{(1i)}ij} + S_{\widetilde{(2i)}ij} - S_{\widetilde{(1i)}i\widetilde{(2i)}} \Big) \\
& \quad \times A_3^0(\widetilde{(1i)}_q, k_g, \widetilde{(2i)}_{\bar q})\, A_3^0(\widetilde{((1i)k)}_q, j_g, \widetilde{((2i)k)}_{\bar q})\, J_3^{(3)}(\widetilde{p_{(1i)k}}, p_j, \widetilde{p_{(2i)k}}) \\
& -\frac{1}{2}\Big(S_{\widetilde{((1k)i)}k\widetilde{((2k)i)}} - S_{\widetilde{((1k)i)}kj} - S_{\widetilde{((2k)i)}kj} + S_{\widetilde{(1k)}kj} + S_{\widetilde{(2k)}kj} - S_{\widetilde{(1k)}k\widetilde{(2k)}} \Big) \\
& \quad \times A_3^0(\widetilde{(1k)}_q, i_g, \widetilde{(2k)}_{\bar q})\, A_3^0(\widetilde{((1k)i)}_q, j_g, \widetilde{((2k)i)}_{\bar q})\, J_3^{(3)}(\widetilde{p_{(1k)i}}, p_j, \widetilde{p_{(2k)i}}) \Bigg\}
\end{aligned}
\tag{3.31}
$$

The new contribution to the N^0 five-parton subtraction term is:

$$
\begin{aligned}
& \mathrm{d}\sigma^A_{NNLO,N^0} = N_5\, N^0\, \mathrm{d}\Phi_5(p_1,\ldots,p_5;q) \\
& \frac{1}{2} \sum_{(i,j)\in(3,4)} \Big(S_{\widetilde{((1i)5)}ij} + S_{\widetilde{((2i)5)}ij} - S_{\widetilde{((1i)5)}i\widetilde{((2i)5)}} - S_{\widetilde{(1i)}ij} - S_{\widetilde{(2i)}ij} + S_{\widetilde{(1i)}i\widetilde{(2i)}} \Big) \\
& \qquad \times A_3^0(\widetilde{(1i)}_q, 5_g, \widetilde{(2i)}_{\bar q})\, A_3^0(\widetilde{((1i)5)}_q, j_g, \widetilde{((2i)5)}_{\bar q})\, J_3^{(3)}(\widetilde{p_{(1i)5}}, \widetilde{p_{(2i)5}}, p_j)
\end{aligned}
\tag{3.32}
$$

These large-angle soft subtraction terms contain soft antenna functions of the form S_{ajc} which is simply the eikonal factor for a soft gluon j emitted between hard partons a and c. Those soft factors are associated with an antenna phase space mapping $(i,j,k) \to (I,K)$. The hard momenta a, c do not need to be equal to the hard momenta i, k in the antenna phase space - they can be arbitrary on-shell momenta.

The integral of each of these soft antenna functions over the antenna phase space can be written as

$$
\begin{aligned}
\mathcal{S}_{ac;ik} &= \int \mathrm{d}\Phi_{X_{ijk}} S_{ajc} \\
&= (s_{IK})^{-\epsilon} \frac{\Gamma^2(1-\epsilon)e^{\epsilon\gamma}}{\Gamma(1-3\epsilon)} \left(-\frac{2}{\epsilon}\right) \left[-\frac{1}{\epsilon} + \ln\left(x_{ac,IK}\right) + \epsilon \,\mathrm{Li}_2\left(-\frac{1-x_{ac,IK}}{x_{ac,IK}}\right)\right] ,
\end{aligned}
\tag{3.33}
$$

where we have defined

$$
x_{ac,IK} = \frac{s_{ac}s_{IK}}{(s_{aI}+s_{aK})(s_{cI}+s_{cK})} \, . \tag{3.34}
$$

So that the integration of the new N^2 subtraction terms reads

$$
\begin{aligned}
\int \mathrm{d}\Phi_{X_{ijk}} \mathrm{d}\sigma^A_{NNLO,N^2} &= N_4\, N^2 \left(\frac{\alpha_s}{2\pi}\right) \mathrm{d}\Phi_4(p_1,\dots,p_4;q) \times \\
& \frac{1}{4} \sum_{(i,j)\in(3,4)} \Bigg\{ \left(\mathcal{S}_{\widetilde{(1i)}\widetilde{(ji)};1j} - \mathcal{S}_{1j;1j} - \mathcal{S}_{2\widetilde{(ji)};1j} + \mathcal{S}_{2j;1j} - \mathcal{S}_{2\widetilde{(1i)};1j} + \mathcal{S}_{12;1j}\right) \\
& \qquad d_3^0(1_q,i_g,j_g)\, A_3^0(\widetilde{(1i)}_q, \widetilde{(ji)}_g, 2_{\bar q}) J_3^{(3)}(\widetilde{p_{1i}}, \widetilde{p_{ji}}, p_2) + (1 \leftrightarrow 2) \\
& \qquad - \left(\mathcal{S}_{\widetilde{(1i)}\widetilde{(2i)};12} - \mathcal{S}_{12;12} - \mathcal{S}_{\widetilde{(2i)}j;12} + \mathcal{S}_{2j;12} - \mathcal{S}_{\widetilde{(1i)}j;12} + \mathcal{S}_{1j;12}\right) \\
& \qquad A_3^0(1_q,i_g,2_{\bar q})\, A_3^0(\widetilde{(1i)}_q, j_g, \widetilde{(2i)}_{\bar q}) J_3^{(3)}(\widetilde{p_{1i}}, p_j, \widetilde{p_{2i}}) \Bigg\}
\end{aligned}
\tag{3.35}
$$

while for the N^0 term the integration of the 5-parton contribution over the antenna phase space yields

$$
\begin{aligned}
\int \mathrm{d}\Phi_{X_{ijk}} \mathrm{d}\sigma^A_{NNLO,N^0} &= N_4\, N^0 \left(\frac{\alpha_s}{2\pi}\right) \mathrm{d}\Phi_4(p_1,p_2,p_3,p_4;q) \\
& \frac{1}{2} \sum_{(i,j)\in(3,4)} \left(\mathcal{S}_{\widetilde{(1i)}j;12} + \mathcal{S}_{\widetilde{(2i)}j;12} - \mathcal{S}_{\widetilde{(1i)}\widetilde{(2i)};12} - \mathcal{S}_{1j;12} - \mathcal{S}_{2j;12} + \mathcal{S}_{12;12}\right)
\end{aligned}
\tag{3.36}
$$

14. Thrust distribution as an example

The terms of the form $\mathrm{d}\sigma^A_{NNLO}$ have now been implemented in the five-parton and four-parton contributions to the N^2 and N^0 colour factors. They lead to changes in the numerical values of the NNLO coefficients which are most pronounced in the approach to the two-jet region. Becher and Schwartz [72] have computed the logarithmically enhanced terms which dominate the thrust distribution in the two-jet region using soft-collinear effective theory. They identified a disagreement with our original numerical results for the thrust distribution [71] in the two-jet region for these two colour factors. Our new results are displayed in figures 9, 10 and 11, and are now in full agreement with the results obtained in [72]. Our numbers also agree with the results obtained in the implementation of [73].

In the genuine three-jet region, which is relevant for precision phenomenology, the changes have a minor numerical impact. The corrections to the NNLO N^2 and N^0 colour factors also affect all other event shape distributions [74] in a similar manner; minor numerical effects in the three-jet region, but more significant effects in the two-jet region.

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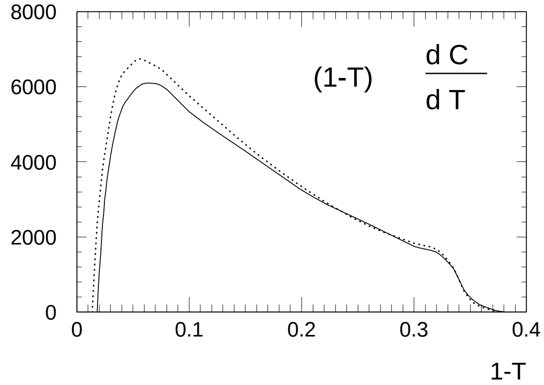

Figure 9: Coefficient of the next-to-next-to-leading order contribution to the thrust distribution. Solid: corrected; dotted: result obtained in [71].

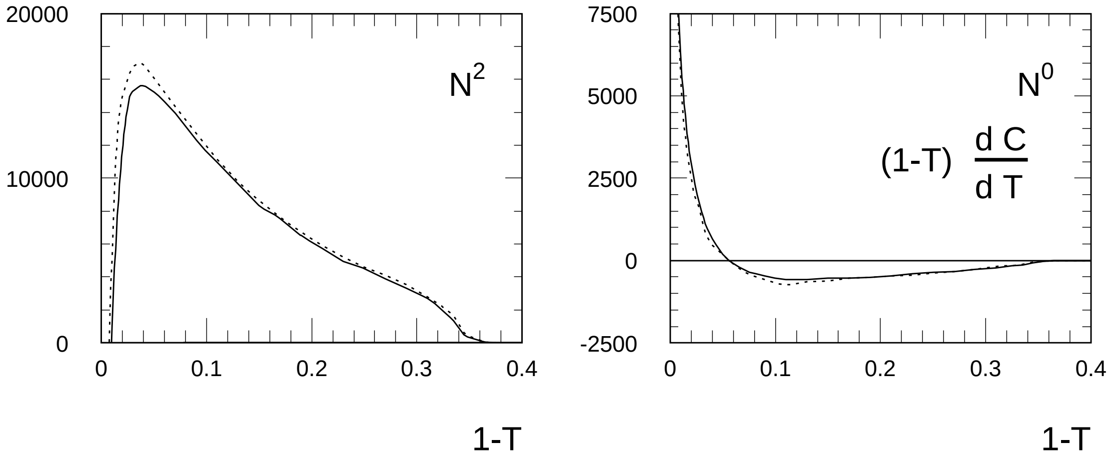

Figure 10: N^2 and N^0 colour factor contributions to NNLO coefficient of the thrust distribution. Solid: corrected; dotted: result obtained in [71].

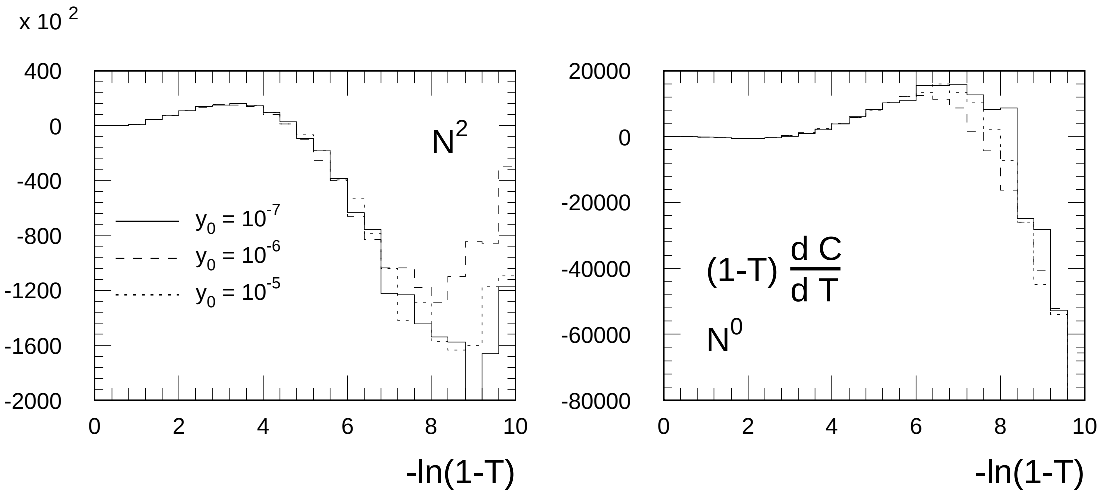

Figure 11: Dependence on phase space cut y_0 in corrected N^2 and N^0 colour factors.

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Added references

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